Quantifying over tuples with Algebra

The Multidimensional Block Product

Klaus-Jörn Lange

September 2015

Contents

Contents

- Logic $\mapsto$ Algebra - The Block Product Lite

Contents

- Logic $\mapsto$ Algebra - The Block Product Lite
- The Counting Case - Sums

The Block Product lite for symmetric quantifiers, only

Contents

- Logic $\mapsto$ Algebra - The Block Product Lite
- The Counting Case - Sums

  The Block Product lite for symmetric quantifiers, only
- The General Case - Sequences

3 Equivalent Views

$$L \subseteq \Sigma^*$$

Existence of constant depth poly-sized circuit family accepting L

$$\Leftrightarrow$$

Existance of first-order formula defining L

3 Equivalent Views

$$L \subseteq \Sigma^*$$

Existence of constant depth poly-sized circuit family accepting L

$$\Leftrightarrow$$

Existance of first-order formula defining L

$$\Leftrightarrow$$

Existence of morphism into a blockproduct recognizing L

$$\phi = Q_1 \, {}_{x_1} Q_2 \, {}_{x_2} \cdots Q_d \, {}_{x_d} \ \ \psi(\vec{x})$$

is transformed into

$$M_\phi = M_{Q_1} \Box M_{Q_2} \cdots \Box M_{Q_d} \Box M_\psi$$

and $h_\phi : \Sigma^* \longrightarrow M_\phi$ such that

$$w \models \phi \Leftrightarrow h_\phi(w) \in M_\phi^+.$$

$$\phi = Q_{1\,x_1} Q_{2\,x_2} \cdots Q_{d\,x_d} \;\; \psi(\vec{x})$$

is transformed into

$$M_\phi = M_{Q_1} \Box M_{Q_2} \cdots \Box M_{Q_d} \Box M_\psi$$

and $h_\phi : \Sigma^* \longrightarrow M_\phi$ such that

$$w \models \phi \Leftrightarrow h_\phi(w) \in M_\phi^+.$$

But this works for unary quantifiers, only!

How to express algebraically $Q_{x_1,x_2,\cdots,x_d}\psi(x_1,\cdots,x_d)$?

How to express algebraically $Q_{x_1,x_2,\cdots,x_d}\psi(x_1,\cdots,x_d)$?

Need to provide sums of the form

$$r_{xy}rr + rr_{xy}r + rrr_{xy} + r_xr_yr + r_xrr_y + r_yr_xr + r_yrr_x + rr_xr_y + rr_yr_x$$

How to express algebraically $Q_{x_1,x_2,\cdots,x_d}\psi(x_1,\cdots,x_d)$?

Need to provide sums of the form

$$r_{xy}rr + rr_{xy}r + rrr_{xy} + r_xr_yr + r_xrr_y + r_yr_xr + r_yrr_x + rr_xr_y + rr_yr_x$$

or sequences/matrices of the form

$$\begin{matrix} r_{xy}rr & r_xr_yr & r_xrr_y \\ r_yr_xr & rr_{xy}r & rr_xr_y \\ r_yrr_x & rr_yr_x & rrr_{xy} \end{matrix}$$

Algebraic Simulation of Logic

Q a first order quantifier, w a word of length $n = |w|$:

Algebraic Simulation of Logic

Q a first order quantifier, w a word of length $n = |w|$:

When is w a model of $Q_x\ \psi(x)$?

$$w \models Q\ x\ \psi(x) \Leftrightarrow$$

Algebraic Simulation of Logic

Q a first order quantifier, w a word of length $n = |w|$:

When is w a model of $Q_x\ \psi(x)$?

$$w \models Q\ x\ \psi(x) \Leftrightarrow$$

First, x is attached to all possible positions

$$\left.\begin{array}{l} w_{x=1} \models \psi? = b_1 \\ w_{x=2} \models \psi? = b_2 \\ \vdots \\ w_{x=i} \models \psi? = b_i \\ \vdots \\ w_{x=n} \models \psi? = b_n \end{array}\right\}$$

Algebraic Simulation of Logic

Q a first order quantifier, w a word of length $n = |w|$:

When is w a model of $Q_x\ \psi(x)$?

$$w \models Q\ x\ \psi(x) \Leftrightarrow$$

First, x is attached to all possible positions

$$\left.\begin{array}{l} w_{x=1} \models \psi? = b_1 \\ w_{x=2} \models \psi? = b_2 \\ \vdots \\ w_{x=i} \models \psi? = b_i \\ \vdots \\ w_{x=n} \models \psi? = b_n \end{array}\right\} \longrightarrow Q$$

Then the resulting values are

evaluated by Q

$Q\text{-Eval}_{1 \le i \le n} b_i$

How to express this in Algebra?

How to express this in Algebra?

Easiest part: express Q by a monoid.

Symmetric Quantifiers

Quantifier	b_+	b_-	Accepting	Rejecting

Symmetric Quantifiers

Quantifier	b_+	b_-	Accepting	Rejecting
$\exists$	$+1$	0	> 0	$= 0$

Symmetric Quantifiers

Quantifier	b_+	b_-	Accepting	Rejecting
$\exists$	$+1$	0	> 0	$= 0$
$\forall$	0	$+1$	$= 0$	> 0

Symmetric Quantifiers

Quantifier	b_+	b_-	Accepting	Rejecting
$\exists$	$+1$	0	> 0	$= 0$
$\forall$	0	$+1$	$= 0$	> 0
$\exists^{\equiv 0(q)}$	1	0	$q \cdot \mathcal{N}$	$q \cdot \mathcal{N} + \{1, 2, \cdots q-1\}$

Symmetric Quantifiers

Quantifier	b_+	b_-	Accepting	Rejecting
$\exists$	$+1$	0	> 0	$= 0$
$\forall$	0	$+1$	$= 0$	> 0
$\exists^{\equiv 0(q)}$	1	0	$q \cdot \mathcal{N}$	$q \cdot \mathcal{N} + \{1, 2, \cdots q-1\}$
Maj	$+1$	-1	> 0	≤ 0

Symmetric Quantifiers

Quantifier	b_+	b_-	Accepting	Rejecting
$\exists$	$+1$	0	> 0	$= 0$
$\forall$	0	$+1$	$= 0$	> 0
$\exists^{\equiv 0(q)}$	1	0	$q \cdot \mathcal{N}$	$q \cdot \mathcal{N} + \{1, 2, \cdots q - 1\}$
Maj	$+1$	-1	> 0	≤ 0

no negative numbers neccessary here

Quantifier Q is expressed by

- M_Q A monoid
- M_Q^+ An accepting subset
- b_+ An element of M_Q representing the value true
- b_- An element of M_Q representing the value false

We are now left with the task to check

A sum in M_Q

$$\left(\sum_{i=1}^{n} \begin{cases} b_+, & w_{x=i} \models \psi \\ b_-, & w_{x=i} \not\models \psi \end{cases} \right) \in M_Q^+?$$

The Symmetric Case

The case M_Q commutative and cyclic

- Dimension 1: Single Variables
- Dimension > 1: Tuples of Variables

The Symmetric Case

The case M_Q commutative and cyclic

- Dimension 1: Single Variables only the half story
- Dimension $>$ 1: Tuples of Variables

The Symmetric Case

The case M_Q commutative and cyclic

- Dimension 1: Single Variables only the half story
- Dimension > 1: Tuples of Variables only 3/4 of the story

The Symmetric Case

The case M_Q commutative and cyclic

- Dimension 1: Single Variables only the half story
- Dimension $>$ 1: Tuples of Variables only 3/4 of the story

Alias:

$M_Q \to B$ **the Base**

$M_\psi \to R$ **the Rest**

ΣR : finite abstract sums (multisets) over R : $\sum_{i \in I} r_i$

Addition in ΣR:

$$\Sigma_{i \in I} r_i + \Sigma_{j \in J} r'_j := \Sigma_{k \in K} r''_k,$$

where K is the disjoint union of I with J and $r''_k = r_k$ if $k \in I$ and $r''_k = r'_k$ otherwise.

Pointwise Multiplication in ΣR:

$$\sum_{i \in I} r_i \cdot \sum_{j \in J} r'_j := \sum_{i \in I, j \in J} r_i r'_j$$

abstract sum of products in R

Neutral element of ΣR is e_R regarded as abstract sum.

The empty sum 0 is a zero: $0 \cdot f = f \cdot 0 = 0$ for all $f \in \Sigma R$ and $0 + f = f + 0 = f$.

Symmetric Construction

The *d*-dimensional case $|\mathcal{V}| = d$

$$\mathcal{N}\boxed{d}R = \left\{ f : 2^{\mathcal{V}} \longrightarrow \Sigma R \mid \ f(\emptyset) \in R \right\}$$

Symmetric Construction

The *d*-dimensional case $|\mathcal{V}| = d$

$$\mathcal{N}\boxed{d}R = \left\{ f : 2^{\mathcal{V}} \longrightarrow \Sigma R \mid \; f(\emptyset) \in R \right\}$$

Multiplication on $\boxed{d}R$: $\quad f, f' \in \boxed{d}R$:

$$f \odot f'(A) =$$

Symmetric Construction

The *d*-dimensional case $|\mathcal{V}| = d$

$$\mathcal{N}\boxed{d}R = \left\{f : 2^{\mathcal{V}} \longrightarrow \Sigma R \mid \; f(\emptyset) \in R\right\}$$

Multiplication on $\boxed{d}R$: $f, f' \in \boxed{d}R$:

$$f \odot f'(A) = \sum_{A=B\cup B', B\cap B'=\emptyset}$$

Symmetric Construction

The d-dimensional case

$|\mathcal{V}| = d$

$$\mathcal{N}\boxed{d}R = \left\{f : 2^{\mathcal{V}} \longrightarrow \Sigma R \mid \ f(\emptyset) \in R\right\}$$

Multiplication on $\boxed{d}R$: $f, f' \in \boxed{d}R$:

$$f \odot f'(A) = \sum_{A=B\cup B', B\cap B'=\emptyset} f(B) \cdot f'(B')$$

Observe $f \odot f'(\emptyset) = f(\emptyset) \cdot f'(\emptyset) \in R$

Symmetric Construction

The *d*-dimensional case $|\mathcal{V}| = d$

$$\mathcal{N}\boxed{d}R = \left\{f : 2^{\mathcal{V}} \longrightarrow \Sigma R \mid \ f(\emptyset) \in R\right\}$$

Multiplication on $\boxed{d}R$: $f, f' \in \boxed{d}R$:

$$f \odot f'(A) = \sum_{A=B\cup B', B\cap B'=\emptyset} f(B)\cdot f'(B')$$

Observe $f \odot f'(\emptyset) = f(\emptyset)\cdot f'(\emptyset) \in R$

Neutral element of $\boxed{d}R$ is f_0 defined by $f_0(\emptyset) := e_R$ and $f_0(A) := 0$ for $A \neq \emptyset$

The Onedimensional Case

Assume $\mathcal{V} = \{x\}$

Write $f \in \boxed{1}R$ as $(r_x := f(\{x\}), r := f(\emptyset))$

The Onedimensional Case

Assume $\mathcal{V} = \{x\}$

Write $f \in \boxed{1}R$ as $(r_x := f(\{x\}), r := f(\emptyset))$

Then $f \odot f = (r_x r + r r_x, rr)$ Our good old block product

The Onedimensional Case

Assume $\mathcal{V} = \{x\}$

Write $f \in \boxed{1}R$ as $(r_x := f(\{x\}), r := f(\emptyset))$

Then $f \odot f = (r_x r + r r_x, rr)$ Our good old block product

and $f \odot f \odot f = (r_x rr + r r_x r + rr r_x, rrr)$

The Twodimensional Case

Assume $\mathcal{V} = \{x, y\}$

Write $f \in \boxed{2}R$ as $\left(r_{xy} := f(\{x, y\}), r_x := f(\{x\}), r_y := f(\{y\}), r := f(\emptyset)\right)$

The Twodimensional Case

Assume $\mathcal{V} = \{x, y\}$

Write $f \in \boxed{2}R$ as $\left(r_{xy} := f(\{x, y\}), r_x := f(\{x\}), r_y := f(\{y\}), r := f(\emptyset)\right)$

Then
$f \odot f = \left(r_{xy}r + rr_{xy} + r_x r_y + r_y r_x, r_x r + rr_x, r_y r + rr_y, rr\right)$ No longer our good old block product!

The Twodimensional Case

Assume $\mathcal{V} = \{x, y\}$

Write $f \in \boxed{2}R$ as $\left(r_{xy} := f(\{x,y\}), r_x := f(\{x\}), r_y := f(\{y\}), r := f(\emptyset)\right)$

Then

$f \odot f = \left(r_{xy}r + rr_{xy} + r_x r_y + r_y r_x, r_x r + r r_x, r_y r + r r_y, rr\right)$ No longer our good old block product!

$f \odot f \odot f(\{x,y\}) = r_{xy}rr + rr_{xy}r + rrr_{xy} + r_x r_y r + r_x r r_y + r_y r_x r + r_y r r_x + r r_x r_y + r r_y r_x$

Evaluation

From $f(\mathcal{V}) = \sum_{i \in I} r_i \in \sum R$ and accepting subset $R^+ \subset R$ build:

Evaluation

From $f(\mathcal{V}) = \sum_{i \in I} r_i \in \sum R$ and accepting subset $R^+ \subset R$ build:

$$\sum_{i \in I} \begin{cases} b_+, & r_i \in R^+ \\ b_-, & r_i \notin R^+ \end{cases}$$

Evaluation

From $f(\mathcal{V}) = \sum_{i \in I} r_i \in \sum R$ and accepting subset $R^+ \subset R$ build:

$$\sum_{i \in I} \begin{cases} b_+, & r_i \in R^+ \\ b_-, & r_i \notin R^+ \end{cases}$$

and then test:

$$\left(\sum_{i \in I} \begin{cases} b_+, & r_i \in R^+ \\ b_-, & r_i \notin R^+ \end{cases} \right) \in B^+?$$

Happy End

• • •

to cut a long story short

$$w \models \phi = Q_{\vec{x}}\psi(\vec{x}) \quad \Leftrightarrow \quad h_\phi(w) \in M_\phi^+ = (M_Q \boxed{d} M_\psi)^+$$

Remarks

$\pi_\emptyset : \boxed{\mathcal{V}} R \longrightarrow R$ defined by $\pi_\emptyset(f) := f(\emptyset)$ is a morphism corresponding to π_2 in the good old block product.

Remarks

$\pi_\emptyset : \boxed{\mathcal{V}} R \longrightarrow R$ defined by $\pi_\emptyset(f) := f(\emptyset)$ is a morphism corresponding to π_2 in the good old block product.

$\pi_\mathcal{V} : \boxed{\mathcal{V}} R \longrightarrow B$ defined by $\pi_\mathcal{V}(f) := B \otimes f(\mathcal{V})$ is not a morphism and somehow corresponds to $\pi_1(.)(e, e)$ in the good old block product.

Remarks

$\pi_\emptyset : \boxed{\mathcal{V}}R \longrightarrow R$ defined by $\pi_\emptyset(f) := f(\emptyset)$ is a morphism corresponding to π_2 in the good old block product.

$\pi_\mathcal{V} : \boxed{\mathcal{V}}R \longrightarrow B$ defined by $\pi_\mathcal{V}(f) := B \otimes f(\mathcal{V})$ is not a morphism and somehow corresponds to $\pi_1(.)(e, e)$ in the good old block product.

Nesting the $\boxed{\mathcal{V}}$-operator needs (finite) formal sums of (finite) formal sums.

The General Case

Case: M_Q not even commutative

- Remarks
- Dimension > 1: Tuples of Variables

The General Case

Case: M_Q not even commutative

- Remarks
- Dimension > 1: Tuples of Variables

Alias:

$M_Q \rightarrow B$ the Base

$M_\psi \rightarrow R$ the Rest

Remarks

Example: oracle quantifier Q^L for $L \subset \{0,1\}^*$

$B = \{0,1\}^*, B^+ + L, b_+ = 1, b_- = 0.$

By convention, quantifiers over tuples are evaluated in lexicographic order, i.e.:

$$Q^L_{x_1,\cdots,x_d}\psi(x_1,\cdots,x_d) \text{ is evaluated as } \prod_{x_1=1}^{n}\cdots\prod_{x_d=1}^{n}\psi(\vec{x})$$

General Construction

The d-dimensional case: Again $\mathcal{V} = \{x_1, x_2, \cdots, x_d\}$

$$\Sigma^{*}\boxed{d}'R = \left\{f = (f_A)_{A\subseteq\mathcal{V}} \mid n \geq 0,\ f_A : \{1, 2, \cdots, n\}^A \longrightarrow R\right\}$$

up to d-dimensional cubes in R where $d = |\mathcal{V}|$. Observe $f_\emptyset \in R$

General Construction

The d-dimensional case: Again $\mathcal{V} = \{x_1, x_2, \cdots, x_d\}$

$$\Sigma^* \boxed{d}' R = \left\{ f = (f_A)_{A \subseteq \mathcal{V}} \mid n \geq 0,\ f_A : \{1, 2, \cdots, n\}^A \longrightarrow R \right\}$$

up to d-dimensional cubes in R where $d = |\mathcal{V}|$. Observe $f_\emptyset \in R$

We set $|f| := n$

General Construction

The d-dimensional case: Again $\mathcal{V} = \{x_1, x_2, \cdots, x_d\}$

$$\Sigma^{*}\boxed{d}'R = \left\{f = (f_A)_{A\subseteq\mathcal{V}} \mid n \geq 0,\ f_A : \{1, 2, \cdots, n\}^A \longrightarrow R\right\}$$

up to d-dimensional cubes in R where $d = |\mathcal{V}|$. Observe $f_\emptyset \in R$

We set $|f| := n$

Multiplication $\odot$ on $\boxed{d}'R$: $f, f' \in \boxed{d}'R$ and $A \subseteq \mathcal{V}$:

General Construction

The d-dimensional case: Again $\mathcal{V} = \{x_1, x_2, \cdots, x_d\}$

$$\Sigma^{*}\boxed{d}'R = \left\{ f = (f_A)_{A\subseteq\mathcal{V}} \mid n \geq 0,\ f_A : \{1,2,\cdots,n\}^A \longrightarrow R \right\}$$

up to d-dimensional cubes in R where $d = |\mathcal{V}|$. Observe $f_\emptyset \in R$

We set $|f| := n$

Multiplication $\odot$ on $\boxed{d}'R$: $f, f' \in \boxed{d}'R$ and $A \subseteq \mathcal{V}$:

$$(f \odot f')_A(i_x)_{x\in A} :=$$

General Construction

The d-dimensional case: Again $\mathcal{V} = \{x_1, x_2, \cdots, x_d\}$

$$\Sigma^* \boxed{d}' R = \left\{ f = (f_A)_{A \subseteq \mathcal{V}} \mid n \geq 0,\ f_A : \{1, 2, \cdots, n\}^A \longrightarrow R \right\}$$

up to d-dimensional cubes in R where $d = |\mathcal{V}|$. Observe $f_\emptyset \in R$

We set $|f| := n$

Multiplication $\odot$ on $\boxed{d}' R$: $f, f' \in \boxed{d}' R$ and $A \subseteq \mathcal{V}$:

$$(f \odot f')_A (i_x)_{x \in A} :=$$

$$1 \leq i_x \leq |f \odot f'| := |f| + |f'|$$

General Construction

The d-dimensional case: Again $\mathcal{V} = \{x_1, x_2, \cdots, x_d\}$

$$\Sigma^{*}\boxed{d}'R = \left\{f = (f_A)_{A\subseteq\mathcal{V}} \mid n \geq 0,\ f_A : \{1,2,\cdots,n\}^A \longrightarrow R\right\}$$

up to d-dimensional cubes in R where $d = |\mathcal{V}|$. Observe $f_\emptyset \in R$

We set $|f| := n$

Multiplication $\odot$ on $\boxed{d}'R$: $f, f' \in \boxed{d}'R$ and $A \subseteq \mathcal{V}$:

$$(f \odot f')_A(i_x)_{x\in A} :=$$

where $B := \{x \in A \mid i_x \leq |f|\}$ and $B' := A \setminus B$, $1 \leq i_x \leq |f \odot f'| := |f| + |f'|$

General Construction

The d-dimensional case: Again $\mathcal{V} = \{x_1, x_2, \cdots, x_d\}$

$$\Sigma^* \boxed{d}' R = \left\{ f = (f_A)_{A \subseteq \mathcal{V}} \mid n \geq 0,\ f_A : \{1, 2, \cdots, n\}^A \longrightarrow R \right\}$$

up to d-dimensional cubes in R where $d = |\mathcal{V}|$. Observe $f_\emptyset \in R$

We set $|f| := n$

Multiplication $\odot$ on $\boxed{d}' R$: $f, f' \in \boxed{d}' R$ and $A \subseteq \mathcal{V}$:

$$(f \odot f')_A (i_x)_{x \in A} := f_B (i_x)_{x \in B} f'_{B'} (i_x - |f|)_{x \in B'}$$

a multiplication in R. Observe: $(f \odot f')_\emptyset \in R$

where $B := \{x \in A \mid i_x \leq |f|\}$ and $B' := A \setminus B$, $1 \leq i_x \leq |f \odot f'| := |f| + |f'|$

General Construction

The d-dimensional case: Again $\mathcal{V} = \{x_1, x_2, \cdots, x_d\}$

$$\Sigma^{*}\boxed{d}'R = \left\{f = (f_A)_{A \subseteq \mathcal{V}} \mid n \geq 0,\ f_A : \{1, 2, \cdots, n\}^A \longrightarrow R\right\}$$

up to d-dimensional cubes in R where $d = |\mathcal{V}|$. Observe $f_\emptyset \in R$

We set $|f| := n$

Multiplication $\odot$ on $\boxed{d}'R$: $f, f' \in \boxed{d}'R$ and $A \subseteq \mathcal{V}$:

$$(f \odot f')_A(i_x)_{x \in A} := f_B(i_x)_{x \in B} f'_{B'}(i_x - |f|)_{x \in B'}$$

a multiplication in R. Observe: $(f \odot f')_\emptyset \in R$

where $B := \{x \in A \mid i_x \leq |f|\}$ and $B' := A \setminus B$, $1 \leq i_x \leq |f \odot f'| := |f| + |f'|$

Neutral element of $\boxed{d}'R$ is ϵ which is the f with $|f| = 0$ and $f_\emptyset = e_R$

The Onedimensional Case

Assume $\mathcal{V} = \{x\}$

Write $f \in \boxed{1}' R$ as $(r_x := f_{\{x\}}, r := f_\emptyset)$

The Onedimensional Case

Assume $\mathcal{V} = \{x\}$

Write $f \in \boxed{1}' R$ as $(r_x := f_{\{x\}}, r := f_\emptyset)$

Then $f \odot f = ((r_x r, r r_x), rr)$ Our good old block product

The Onedimensional Case

Assume $\mathcal{V} = \{x\}$

Write $f \in \boxed{1}' R$ as $(r_x := f_{\{x\}}, r := f_\emptyset)$

Then $f \odot f = ((r_x r, r r_x), rr)$ Our good old block product

And $f \odot f \odot f = ((r_x rr, r r_x r, rr r_x), rrr)$

The Twodimensional Case

Assume $\mathcal{V} = \{x, y\}$

Write $f \in \boxed{2}' R$ as $\left(r_{xy} := f_{\{x,y\}}, r_x := f_{\{x\}}, r_y := f_{\{y\}}, r := f_\emptyset\right)$ $|f| = 1$

Then

The Twodimensional Case

Assume $\mathcal{V} = \{x, y\}$

Write $f \in \boxed{2}' R$ as $\left(r_{xy} := f_{\{x,y\}}, r_x := f_{\{x\}}, r_y := f_{\{y\}}, r := f_\emptyset\right)$ $|f| = 1$

Then $f \odot f = \left(\begin{matrix} r_{xy} r & r_x r_y \\ r_y r_x & r r_{xy} \end{matrix}, \begin{matrix} r_x r \\ r r_x \end{matrix}, (r_y r, r r_y), rr \right)$ No longer our good old block product!

The Twodimensional Case

Assume $\mathcal{V} = \{x, y\}$

Write $f \in \boxed{2}'R$ as $\left(r_{xy} := f_{\{x,y\}}, r_x := f_{\{x\}}, r_y := f_{\{y\}}, r := f_\emptyset\right)$ $|f| = 1$

Then $f \odot f = \left(\begin{matrix} r_{xy}r & r_x r_y \\ r_y r_x & r r_{xy} \end{matrix} , \begin{matrix} r_x r \\ r r_x \end{matrix} , (r_y r, r r_y), rr \right)$ No longer our good old block product!

$$((f \odot f) \odot f)_{\{x,y\}} = \begin{array}{cc|c} r_{xy}rr & r_x r_y r & r_x r r_y \\ r_y r_x r & r r_{xy} r & r r_x r_y \\ \hline r_y r r_x & r r_y r_x & r r r_{xy} \end{array} =$$

$$\begin{array}{c|cc} r_{xy}rr & r_x r_y r & r_x r r_y \\ \hline r_y r_x r & r r_{xy} r & r r_x r_y \\ r_y r r_x & r r_y r_x & r r r_{xy} \end{array} = (f \odot (f \odot f))_{\{x,y\}}$$

Happy End

• • •

to cut a long story short

$$w \models \phi = Q_{\vec{x}}\psi(\vec{x}) \quad \Leftrightarrow \quad h_\phi(w) \in M_\phi^+ = (M_Q \boxed{d} M_\psi)^+$$

Discussion

- Construction works only for strong block product, not for weak one?

Discussion

- Construction works only for strong block product, not for weak one?
- Thanks for your patience!