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PROOF-THEORETIC HARMONY: THE ISSUE OF PROPOSITIONAL QUANTIFICATION

Abstract We present our calculus of higher-level rules, extended with propositional quantification within rules. This makes it possible to present general schemas for introduction and elimination rules for arbitrary propositional operators and to define what it means that introductions and eliminations are in harmony with each other. This definition does not presuppose any logical system, but is formulated in terms of rules themselves. We therefore speak of a foundational (rather than reductive) account of proof-theoretic harmony. With every set of introduction rules a canonical elimination rule, and with every set of elimination rules a canonical introduction rule is associated in such a way that the canonical rule is in harmony with the set of rules it is associated with. An example given by Hazen and Pelletier is used to demonstrate that there are significant connectives, which are characterized by their elimination rules, and whose introduction rule is the canonical introduction rule associated with these elimination rules. Due to the availabiliy of higher-level rules and propositional quantification, the means of expression of the framework developed are sufficient to ensure that the construction of canonical elimination or introduction rules is always possible and does not lead out of this framework[1].

1 The full text of this paper is published under the title "Higher-level rules, propositional quantification, and the foundational approach to proof-theoretic harmony" in Studia Logica (Special issue, ed. Andrzej Indrzejczak, commemorating the 80th anniversary of Gentzen's and Jaśkowski's groundbreaking works on assumption based calculi). This work was carried out within the French-German ANR-DFG project "Hypothetical Reasoning — Its Proof-Theoretic Analysis" (HYPOTHESES) (DFG Schr 275/16-2).

1 INTRODUCTION

Both Gentzen's calculus of natural deduction ([7]) and Jaśkowski's calculus of suppositions ([9]) are based on what may be called the dynamic view of assumptions. According to the traditional static view, assumptions are suppositions on which all subsequent formulas in a derivation depend. According to the dynamic view assumptions made (or 'introduced') can be discharged (or 'eliminated') at the application of certain rules. The dynamic view of assumptions makes it possible to give implication a proof-theoretic meaning based on the idea that asserting an implication means the same as deriving its consequent from its antecedent.

This paper deals with the systematics of introduction and elimination rules in natural deduction and their relationship often described as 'harmony' ([5]) and is therefore related to Gentzen's approach. However, it does so by using the idea of rules of higher levels, which extends the dynamic view of assumptions by not only allowing that assumptions be discharged but also that assumptions be introduced in the course (and not only at the top) of a derivation, where these assumptions are not necessarily formulas, but can be rules as well. This idea is here extended to include propositional quantification within rules, i.e. the idea that rules may universally quantify over propositions.

The idea to study propositional quantification occurs already in Jaśkowski's paper on suppositions ([9]), but not in Gentzen's work on natural deduction. Jaśkowski studies propositional quantifiers before he passes on to first-order ones. However, unlike Jaśkowski, we do not use propositional quantification and propositional eigenvariables in rules in order to define propositional quantifiers, but in order to define propositional connectives. Thus one of our central claims is that propositional quantification is useful and indeed necessary to study propositional connectives from a general proof-theoretic perspective. This claim comes along with the view that introduction rules are not given priority over elimination rules, as in Gentzen's work, but that introductions and eliminations are treated on par. In fact, it will be the elimination-based approach where the idea of propositional quantification in rules develops its full power.

In ([14]) a notion of proof-theoretic harmony was proposed. There the meaning of a connective according to given introduction rules was

described by a formula of second-order intuitionistic propositional logic PL2, and likewise for eliminaton rules. When the two formulas obtained were equivalent, introduction and elimination rules were said to be in harmony. This approach was called **reductive**, since it took the system PL2 for granted, which means that it did not apply to operators such as conjunction, disjunction or implication, as they are already an ingredient of PL2[2]. Now we follow a **foundational** approach in that we shall define a notion of harmony between introduction and elimination rules which is exclusively defined in terms of the rules used rather than in terms of certain formulas of an external system.

Following the idea presented in [13], we shall define a purely structural calculus of rules, which is considered more elementary than any logical system, where these rules may now contain propositional quantification to express generality. If we just want to express, for example, that for any A and B, $A \wedge B$ can be inferred from A and B, we do not necessarily need propositional quantification. We can just use A and B as schematic letters in the rule schema

$$\frac{A \quad B}{A \wedge B} \ .$$

However, if we want to express that $\neg A$ can be inferred whenever, for any B, A entails B, we need some propositional variable-binding device. In the notation to be developed, we write such a rule two-dimensionally as

$$\frac{\begin{pmatrix} A \\ p \end{pmatrix}_p}{\neg A}$$

or linearly as

$$(A \Rightarrow_p p) \Rightarrow \neg A \ .$$

Formally this means that, in order to infer $\neg A$, it is sufficient to derive p from A and possibly further assumptions, where p must not occur free in A or any other assumption on which p depends.

2 Since conjunction and disjunction are definable in PL2, only implication and propositional quantification are actually needed.

When developing a general schema for elimination rules given certain introduction rules, no quantification is needed. If the introduction rules do not contain any quantification, neither does the general elimination rule, as quantification from outside can be expressed by schematic variables. However, when developing a general schema for introduction rules given certain elimination rules, we need this sort of quantification. Even without such a general schema, when defining harmony for arbitrary introduction and elimination rules, the availability of propositional quantification is crucial. Propositional quantification is a very elementary device in rule application, as it essentially relies on the proper handling of variables. However, it gives us powerful new structural means of expression.

When using second-order quantification, we are exposed to the objection of employing impredicative notions. It should be noted that our formulas do not contain any quantifiers, as quantification occurs only in rules and not in formulas. Of course, the quantified propositional variables run over propositional formulas which may contain connectives which are defined by using these rules. However, if one calls this way of defining a propositional connective impredicative, then it is impredicativity of a harmless sort. Then even the standard rule for disjunction elimination

$$\dfrac{\begin{array}{ccc} & [A] & [B] \\ A\vee B & C & C \end{array}}{C}$$

would be impredicative, as the schematic letter C runs over arbitrary formulas, and thus in particular over $A\vee B$ itself. This sort of impredicativity is harmless, as we are using variables only in a schematic sense, corresponding to what Carnap in [2] called "specific" (in contradistinction to "numeric") generality. The handling of quantified rules is described by the proper handling of variables in derivations.

2 HIGHER-LEVEL RULES WITH PROPOSITIONAL QUANTIFICATION

Our calculus of higher-level rules is essentially a pure calculus of suppositions, where the suppositions are of an extended form. Unlike

the presentation in [13], where rules were identified with the schema of their application, we here provide a more elementary approach in which rules are expressions labelling their application in a derivation, and are not just extracted from a certain inference figure. The general form of a rule is

$$(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow_{\overrightarrow{p}} A\,, \tag{1}$$

where $n \geq 0$ and $\overrightarrow{q_1}, \ldots, \overrightarrow{q_n}, \overrightarrow{p}$ are (possibly empty) lists of propositional variables. The variables occurring as indices to the rule arrow $\Rightarrow$ are bound in the premisses and the conclusion of the rule, so that the usual restrictions concerning substitutions apply. The intended meaning of a rule of form (1) is the following: **For any $\overrightarrow{p}$: Suppose, for each i $(1 \leq i \leq n)$, we have derived B_i from Γ_i, where this derivation is schematic in $\overrightarrow{q_i}$; then we may pass over to A.** That the derivation of B_i from Γ_i is schematic in $\overrightarrow{q_i}$ is expressed by an eigenvariable condition. That the rule can be applied for any $\overrightarrow{p}$ is expressed by allowing for arbitrary substitutions of lists of formulas for $\overrightarrow{p}$. According to this reading the variables occurring as indices to the rule arrow $\Rightarrow$ function as universal quantifiers. If such variables are present, we speak of **quantified** (higher-level) rules, or of (higher-level) rules **with quantification**. Formally the intended meaning of a rule is explained by giving a schema according to which a rule of the form (1) is applied in a derivation. Note that when higher-level rules are available as assumptions, there is no need for a formal system to contain primitive rules of inference, as the assumption rules already provide the means to generate formulas from others. For example,

$$\dfrac{\dfrac{\overline{q_2}\,q_2 \quad \overline{D_2}\,D_2}{C}\,[q_2, D_2 \Rightarrow C]^1}{\vdots}$$
$$1\,\dfrac{\overline{B_1}\,B_1 \qquad q_2}{A}\,q_1, ((q_2, D_2 \Rightarrow r) \Rightarrow_{q_2} q_2) \Rightarrow_{pq_1r} p$$

is a derivation of A from the assumptions B_1, q_2, D_2 and the assumption rule $q_1, ((q_2, D_2 \Rightarrow r) \Rightarrow_{q_2} q_2) \Rightarrow_{pq_1r} p$, where at the application of this latter assumption rule the assumption rule $q_2, D_2 \Rightarrow C$

is discharged (as indicated by the numeral). Therefore, as it does not use any primitive rules of inference, this derivation is purely structural in our sense. For primitive rules of inference we also use a two-dimensional notation, which is often better readable than the 'official' one-dimensional notation. Instead of

$$(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow_{\overrightarrow{p}} A,$$

where $\overrightarrow{p}$ comprises **all** variables free in $(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow A$, we also write:

$$\frac{\begin{pmatrix}\Gamma_1\\ B_1\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Gamma_n\\ B_n\end{pmatrix}_{\overrightarrow{q_n}}}{A}$$

where the parentheses can be omitted when $\overrightarrow{q_n}$ is empty. Our proviso concerning the variables $\overrightarrow{p}$ means in effect that at the inference line all variables free above or below become bound. In other words, we only consider primitive inference rules without free variables. This convention is fully appropriate and sufficient for our aims.

One crucial philosophical point of our dealing with rules is the fact that rules are always **applied** in a derivation. They never occur as items that are **asserted**. Only formulas can be asserted. This **applicative behavior** is what makes rules a most fundamental entity, whose usage can be explained without recurring to logic and therefore can serve in a foundational approach to logic.

3 INTRODUCTIONS AND ELIMINATIONS

Using this extended notion of inference rules we can propose general notions of introduction and elimination rules. Introductions and eliminations are considered independent kinds rules which are not expected to stand in any particular relation to each other. An introduction rule for

an n-ary propositional operator c, where $\overrightarrow{p}$ represents its arguments $p_1, \ldots, p_n$, has the general form

$$\frac{\begin{pmatrix}\Gamma_1\\ B_1\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Gamma_m\\ B_m\end{pmatrix}_{\overrightarrow{q_m}}}{c(\overrightarrow{p})} \qquad (2)$$

,

and an elimination rule is of the form

$$\frac{c(\overrightarrow{p}) \quad \begin{pmatrix}\Gamma_1\\ B_1\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Gamma_\ell\\ B_\ell\end{pmatrix}_{\overrightarrow{q_\ell}}}{C} \qquad (3)$$

.

We define a **canonical** elimination rule given arbitrary introduction rules, and a **canonical** introduction rule given arbitrary elimination rules, which play a special role. The canonical elimination rule for given introduction rules is the uniform general elimination rule proposed in [13]. The canonical introduction rule for given elimination rules is a uniform general introduction rule. If the following list of introduction rules is associated with c:

$$\begin{cases} \Delta_1 \quad \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q_1}} \quad c(\overrightarrow{p}) \\ \qquad\quad \vdots \\ \Delta_k \quad \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q_k}} \quad c(\overrightarrow{p}) \quad , \end{cases} \qquad (4)$$

then the canonical elimination rule has the form

$$c(\overrightarrow{p}), (\Delta_1 \Rightarrow_{\overrightarrow{q_1}} r), \ldots, (\Delta_k \Rightarrow_{\overrightarrow{q_k}} r) \;\Rightarrow_{\overrightarrow{p}\,r}\; r\ , \qquad (5)$$

or two-dimensionally:

$$\frac{c(\overrightarrow{p}) \quad \begin{pmatrix}\Delta_1\\ r\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Delta_k\\ r\end{pmatrix}_{\overrightarrow{q_k}}}{r} \qquad (6)$$

.

where r is a fresh variable not occurring in $\overrightarrow{p}, \overrightarrow{q_1}, \ldots, \overrightarrow{q_k}$. If the following list of elimination rules is associated with with c:

$$\left\{\begin{array}{lll} c(\overrightarrow{p}), \Delta_1' & \Rightarrow_{\overrightarrow{q_1}\overrightarrow{p}} & C_1 \\ & \vdots & \\ c(\overrightarrow{p}), \Delta_{k'}' & \Rightarrow_{\overrightarrow{q_{k'}}\overrightarrow{p}} & C_{k'} \quad , \end{array}\right. \tag{7}$$

then the canonical introduction rule has the form

$$(\Delta_1' \Rightarrow_{\overrightarrow{q_1}} C_1), \ldots, (\Delta_{k'}' \Rightarrow_{\overrightarrow{q_{k'}}} C_{k'}) \;\Rightarrow_{\overrightarrow{p}} c(\overrightarrow{p}) \;, \tag{8}$$

or two-dimensionally:

$$\frac{\begin{pmatrix}\Delta_1' \\ C_1\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Delta_{k'}' \\ C_{k'}\end{pmatrix}_{\overrightarrow{q_{k'}}}}{c(\overrightarrow{p})} \tag{9}$$

.

4 HARMONY

Harmony is now defined in such a way that both a 'no-gain' and a 'no-loss' criterion is fulfilled. 'No-gain' means that an introduction followed by an elimination does not give us anything new, i.e., such a sequence of steps can be removed. This criterion may also be called 'local reduction' and is related to Belnap's criterion of convervativeness ([1]). 'No-loss' means that the consequences of eliminations suffice to restitute their major premiss. This criterion is also called 'criterion of recovery', and is related to Belnap's criterion of uniqueness and Dummett's notion of stability ([4]). Formally, given introductions and eliminations of the forms 4 and 7, respectively, 'no-gain' says that we can derive C_i from any Δ_i, Δ_j' without using primitive rules of inference, i.e. that

$$\Delta_i, \Delta_j' \vdash_{\{struc\}} C_i \quad \text{for all } i, j$$

holds as a purely structural derivability assertion. 'No-loss' says that c-introduction rules alone suffice to establish $c(\overrightarrow{p})$, given the consequences of the eliminations, i.e.,

$$(\Delta'_1 \Rightarrow_{\overrightarrow{q_1}} C_1), \ldots, (\Delta'_{k'} \Rightarrow_{\overrightarrow{q_{k'}}} C_{k'}) \vdash_{\{I-rules\}} c(\overrightarrow{p}) \ .$$

These criteria are defined for any set of introductions and eliminations given. It can then be shown that the canonical elimination rule 5/6 is in harmony with its associated introductions 4, and that the canonical introduction rule 8/9 is in harmony with its associated eliminations 7. To avoid any misinterpretation of this result, it should be repeated that the notion of harmony itself is **not** based on the construction of general canonical elimination or general canonical introduction rules, but rather on the independent schemas 2 and 3. This makes our approach different from other approaches which base harmony on the canonical form of rules (normally elimination rules[3]).

5 EXAMPLE

It is crucial that we use propositional quantification. The canonical introduction rule cannot be formulated if no propositional quantification is derivable (or more precisely, it could then only be formulated for very simple cases of elimination rules). The fact that the consideration of a general canonical introduction rule for given elimination rules is not just a theoretical possibility dealt with for reasons of symmetry, but has an intrinsic value, is demonstrated by using an example provided by Hazen and Pelletier ([8]). It defines a theoretically significant (namely expressively complete) ternary connective ★ in terms of its elimination rules, while its harmonious introduction is the canonical introduction rule. ★ has the meaning $(p_1 \vee p_2) \leftrightarrow (p_3 \leftrightarrow \neg p_2)$, a formula, of which Došen in [3] could show that it represents a Sheffer (i.e., expressively

3 Such as the programme described by Read as 'general-elimination-harmony', see [12, 11, 6].

complete) connective for intuitionistic propositional logic. Its elimination rules derived by Hazen and Pelletier from this meaning are

$$\frac{\bigstar(p_1,p_2,p_3) \quad p_1 \quad \begin{pmatrix} p_2 \\ q \end{pmatrix}_q}{p_3} \qquad \frac{\bigstar(p_1,p_2,p_3) \quad p_2 \quad p_3}{q}$$

$$\frac{\bigstar(p_1,p_2,p_3) \quad \begin{pmatrix} p_2,p_3 \\ q \end{pmatrix}_q \quad \begin{matrix} p_2 \Rightarrow_q q \quad p_1 \quad p_2 \\ p_3 \qquad\quad r \qquad r \end{matrix}}{r}$$

while its introduction rule given by Hazen and Pelletier is nothing but the canonical introduction rule in our sense:

$$\frac{\begin{matrix} p_1,(p_2 \Rightarrow_q q) \\ p_3 \end{matrix} \quad \begin{pmatrix} p_2,p_3 \\ q \end{pmatrix}_q \quad \begin{pmatrix} (p_2,p_3 \Rightarrow_q q),(p_2 \Rightarrow_q q) \Rightarrow p_3),(p_1 \Rightarrow r),(p_2 \Rightarrow r) \\ r \end{pmatrix}_r}{\bigstar(p_1,p_2,p_3)}.$$

6 NEGATIVE RESULTS

Though for every list of given introduction rules there is a canonical elimination rule, and for every list of given elimination rules there is a canonical introduction rule, this canonical rule is always of a level higher than the maximum level of its corresponding introduction or elimination rules. This is due to the fact that the premisses of introductions become dischargeable assumptions in the canonical elimination, and the premisses of eliminations become dischargeable assumptions in the canonical introduction. For example, the connective $\star$ with the introduction rules

$$\frac{\begin{matrix} p_1 \\ p_2 \end{matrix}}{\star(p_1,p_2,p_3)} \qquad \frac{p_3}{\star(p_1,p_2,p_3)}$$

of maximum level 2 receives a canonical elimination rule

$$\dfrac{\star(p_1,p_2,p_3) \qquad \begin{matrix} p_1 \Rightarrow p_2 & p_3 \\ q & q \end{matrix}}{q}$$

of level 3, and the connective $\circ$ with the elimination rule

$$\dfrac{\circ(p_1,p_2,p_3) \qquad \begin{matrix} p_1 \\ p_2 \end{matrix}}{p_3}$$

of level 2 receives a canonical introduction rule

$$\dfrac{\begin{matrix} p_1 \Rightarrow p_2 \\ p_3 \end{matrix}}{\circ(p_1,p_2,p_3)}$$

of level 3. In [10] it is shown that this behavior cannot be avoided, i.e. that there are no harmonious elimination rules for $\star$ and no harmonious introduction rules for $\circ$ of lower level.

7 GENERAL CONCLUSION

As soon as higher-level rules and propositional quantification are available, the generation of canonical eliminations and introductions from given introductions and eliminations, respectively, does not generate novel sorts of rules. This is a remarkable closure property with respect to the expressive power of introduction and elimination rules.

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