Three Lectures on Dialogues

by

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Preface

These are the lecture notes to three lectures on dialogical logic given at the *Institut d'Histoire et de Philosophie des Sciences et des Techniques* in Paris on March 14, 21 and 28, 2013. I would like to thank all the students for their participation, and I would like to thank Jean Fichot for his kind invitation. The lectures were supported by the French-German ANR-DFG project "Hypothetical Reasoning – Its Proof-Theoretic Analysis" (HYPOTHESES). The lecture notes themselves are mainly based on Piecha [2012] and Piecha and Schroeder-Heister [2012]. Results are given without proofs here; the reader can find them in Piecha [2012].

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Contents

First Lecture: An Introduction to Dialogues

1.1 Introduction

Dialogues have first been proposed by Lorenzen [1960, 1961] as an alternative foundation for constructive or intuitionistic logic. The general idea is that the logical constants are given an interpretation in certain game-theoretical terms. Dialogues are two-player games between a proponent and an opponent, where each of the two players can either attack claims made by the other player or defend their own claims. For example, an implication $A \to B$ is attacked by claiming A and defended by claiming B. This means that in order to have a winning strategy for $A \to B$, the proponent must be able to generate an argument for B depending on what the opponent can put forward in defense of A. The logical constant of implication has thus been given a certain game-theoretical or argumentative interpretation, and corresponding argumentative interpretations can be given for the other logical constants as well.

In this first lecture we will learn about dialogues for intuitionistic logic.

In the second lecture we will consider definitional dialogues that allow us to reason about given definitions.

In the third lecture we will explain dialogues for implications considered as rules.

1.2 Dialogues and strategies for propositional logic

We define the concepts of argumentation form, dialogue and strategy, following the presentation of Felscher [1985, 2002] with slight deviations. We focus on dialogues for intuitionistic propositional logic. In contradistinction to classical logic, the law of excluded middle (*tertium non datur*) $A \vee \neg A$ does *not* hold in intuitionistic logic, and implication is a genuine logical constant which cannot be expressed by using negation and disjunction (i.e. “$A \to B \neq \neg A \vee B$”).

1.2.1 Dialogues

We define our language, argumentation forms for logical constants and dialogues.

Definition 1.1 (i) The *language* consists of propositional *formulas* A, B, $\ldots, A_1, \ldots$ that are constructed from *atomic formulas* (*atoms*) $a, b, \ldots$, $a_1, \ldots$ with the *logical constants* $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication) and $\neg$ (negation). *language*

(ii) Furthermore, $\wedge_1$, $\wedge_2$ and $\vee$ are used as *special symbols*. *special symbols*

(iii) In addition, the *signatures* P ('proponent') and O ('opponent') are used. *signatures*

(iv) An *expression* e is either a formula or a special symbol. For each expression e there is a P*-signed expression* $P\,e$ and an O*-signed expression* $O\,e$. *expression*

(v) A signed expression is called *assertion* if the expression is a formula; it is called *symbolic attack* if the expression is a special symbol. X and Y, where $X \neq Y$, are used as variables for P and O. *assertion* *symbolic attack*

Definition 1.2 For each logical constant an *argumentation form* determines how a complex formula (having the respective constant in outermost position) that is asserted by X can be attacked by Y and how this attack can be defended (if possible) by X. The argumentation forms are as follows: *argumentation form*

conjunction $\wedge$:	assertion:	$X\,A_1 \wedge A_2$	
	attack:	$Y \wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X\,A_i$	
disjunction $\vee$:	assertion:	$X\,A_1 \vee A_2$	
	attack:	$Y \vee$	
	defense:	$X\,A_i$	(X chooses $i = 1$ or $i = 2$)
implication $\to$:	assertion:	$X\,A \to B$	
	attack:	$Y\,A$	
	defense:	$X\,B$	
negation $\neg$:	assertion:	$X\,\neg A$	
	attack:	$Y\,A$	
	defense:	*no defense*	

Example 1.1 The following is a concrete instance of the argumentation form for implication:

$$P\,\neg a \to (b \vee a)$$
$$O\,\neg a$$
$$P\,b \vee a$$

Remark 1.2 By these argumentation forms the logical constants are given an *argumentative interpretation* (as Felscher [2002, p. 127] calls it) in the following sense: *argumentative interpretation*

(i) An argument on a conjunctive assertion made by X consists in Y choosing one conjunct of the assertion, and X continuing the argument with that chosen conjunct. In other words, the argumentative interpretation of conjunction is given by the reduction of the argument on a conjunctive assertion made by X to the argument on one of the conjuncts chosen by Y in the attack.

(ii) In an argument on a disjunctive assertion made by X, Y demands the continuation of the argument with any of the disjuncts. In other words, the argumentative interpretation of disjunction is given by the reduction of the argument on a disjunctive assertion made by X to the argument on one of the disjuncts chosen by X in the defense.

(iii) An argument on an implicative assertion made by X consists in Y stating the antecedent of the implication (whereby the antecedent functions as an assumption), and X continuing the argument with the succedent. Alternatively, X could continue with an attack on the assumed antecedent. In other words, the argumentative interpretation of implication is given by the reduction of the argument on an implicative assertion made by X to the argument on the succedent under the assumption of the antecedent.

(iv) An argument on a negative assertion $\neg A$ made by X consists in Y stating the assertion A, without X being able to continue the argument.

This argumentative interpretation of negation can be made clear by introducing the *falsum* $\bot$ as a constant which signifies absurdity (which is taken as a primitive notion). We can then define negation by implication and *falsum*: $\neg A := A \to \bot$. An argument on $\neg A$ is thus an argument on $A \to \bot$. However, X asserting $\bot$ would mean that Y could continue the argument with *any* assertion—assuming the principle of *ex falso quodlibet* to be applicable here. To avoid this, $\bot$ must not be asserted. Hence, an argument on $\neg A$ (i.e. on $A \to \bot$) can only continue with an argument on the assumption A, and cannot be reduced to an argument on $\bot$.

This is similar to the treatment of negation in constructive semantics, respectively in the Brouwer–Heyting–Kolmogorov (BHK) interpretation of logical constants, as for example stated by Heyting [1971, p. 102]: "[...] $\neg \mathfrak{p}$ can be asserted if and only if we possess a construction which from the supposition that a construction $\mathfrak{p}$ were carried out, leads to a contradiction." Where contradiction—or equivalently absurdity (here signified by $\bot$)—is usually considered to be a primitive notion.

Definition 1.3 (i) Let $\delta(n)$, for $n \geq 0$, be a signed expression and $\eta(n)$ a pair $[m, Z]$, for $0 \leq m < n$, where Z is either A (for 'attack') or D (for 'defense'), and where $\eta(0)$ is empty. Pairs $\langle \delta(n), \eta(n) \rangle$ are called *moves*. *move*

(ii) A move $\langle \delta(n), \eta(n) = [m, A] \rangle$ is called *attack move*, and a move $\langle \delta(n), \eta(n) = [m, D] \rangle$ is called *defense move*. *attack move* *defense move*

Remark 1.3 $\delta(n)$ is a function mapping natural numbers $n \geq 0$ to signed expressions $X\,e$, and $\eta(n)$ is a function mapping natural numbers $n \geq 0$ to pairs $[m, Z]$. The numbers in the domain of $\delta(n)$ (resp. in the domain of $\eta(n)$) are called *positions*. *positions*

When talking about a move $\langle \delta(n), \eta(n) \rangle$, we write $\langle \delta(n) = X\,e, \eta(n) = [m, Z] \rangle$ to express that $\delta(n)$ has the value $X\,e$ for position n, and that $\eta(n)$ has the value $[m, Z]$ for position n. For example, $\langle \delta(n) = P\,A, \eta(n) = [m, D] \rangle$ denotes a defense move which is made by the proponent P at position n by asserting the formula A; this defense move refers to a move made at position m. A concrete move like $\langle \delta(4) = P \wedge_1, \eta(4) = [3, A] \rangle$ will also be written as

$$4. \quad P \wedge_1 \;\; [3, A]$$

This is an attack move with symbolic attack $P \wedge_1$; it is made at position 4 and refers to a move made at position 3.

The notation $\langle \delta(n) = X\,e, \eta(n) = [m, Z] \rangle$ has the advantage that we can speak about a move $\langle X\,e, [m, Z] \rangle$ by including information about the position n at which this move is made.

Although moves are always pairs $\langle \delta(n), \eta(n) \rangle$, we will also refer to moves by giving only their $\delta(n)$-component, as long as it is clear from the context which move is meant, or if it is irrelevant whether the move is an attack or a defense, or if it is irrelevant to which position the move refers to. And instead of $\langle \delta(n) = X\,e, \eta(n) \rangle$ we will also speak of the move $X\,e$ made at position n. We will also speak simply about attacks and defenses in order to refer to attack moves and defense moves, respectively.

Definition 1.4 A *dialogue* is a finite or infinite sequence of moves $\langle \delta(n), \eta(n) \rangle$ (for $n = 0, 1, 2, \ldots$) satisfying the following conditions: *dialogue*

(D00) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

(D01) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

(D02) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

Definition 1.5 An attack $\langle \delta(n), \eta(n) = [m, A] \rangle$ at position n on an assertion at position m is called *open at position* k for $n < k$ if there is no position n' such that $n < n' \leq k$ and $\langle \delta(n'), \eta(n') = [n, D] \rangle$, that is, if there is no defense at or before position k to an attack at position n. *open attack*

Remark 1.4 Since there is no defense to an attack $\langle \delta(n) = Y\,A, \eta(n) = [m, A] \rangle$ on $\delta(m) = X\,\neg A$ for $m < n$, the attack at position n is open at all positions k for $n < k$.

1.2.2 DI^p-dialogues

We define DI^p-dialogues and strategies. With regard to the literature on dialogical logic, DI^p-dialogues can be considered to be the standard dialogues for intuitionistic propositional logic. The following definition of DI^p-dialogues is based on the definition of dialogues.

Definition 1.6 A DI^p*-dialogue* is a dialogue satisfying the following conditions (in addition to (D00), (D01) and (D02)): DI^p*-dialogue*

(D10) If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$.

That is, P may assert an atomic formula only if it has been asserted by O before.

(D11) If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.

That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

($D12$) For every m there is at most one n such that $\eta(n) = [m, D]$.

That is, an attack may be defended at most once.

($D13$) If m is even, then there is at most one n such that $\eta(n) = [m, A]$.

That is, a P-signed formula may be attacked at most once.

A DI^p-dialogue beginning with $P\,A$ (i.e., $\delta(0) = P\,A$, where A is a complex formula) is called *DI^p-dialogue for the formula A*.

Remark 1.5 The objects defined by the conditions ($D00$)–($D02$) alone are what Felscher [1985, 2002] calls 'dialogues', and the objects defined by adding ($D10$)–($D13$)—which we call 'DI^p-dialogues'—are called 'D-dialogues' by him. Since here we are concerned with the objects defined by ($D00$)–($D02$) plus ($D10$)–($D13$), we simply speak of 'dialogues', omitting the specifier 'DI^p' as long as no confusion can arise. *(dialogues)*

Remark 1.6 The conditions ($D00$)–($D13$) are also called 'structural rules', 'frame rules' ('Rahmenregeln') or 'special rules of the game' ('spezielle Spielregeln') in the literature, and ($D10$) is sometimes called 'formal rule'. The argumentation forms are also called 'particle rules' ('Partikelregeln'), 'logical rules' or 'general rules of the game' ('allgemeine Spielregeln').

We will stick to the notions 'dialogue condition(s)' (or just 'condition(s)') and 'argumentation form(s)'. *(dialogue conditions)*

Remark 1.7 Proponent P and opponent O are not interchangeable due to the asymmetries between P and O introduced by ($D10$) and ($D13$). For atomic formulas a, the proponent move $\langle \delta(n) = P\,a, \eta(n) = [m, Z] \rangle$ is possible only after an opponent move $\langle \delta(m) = O\,a, \eta(m) = [k, Z] \rangle$ for $k < m < n$, and O can attack a P-signed formula only once, whereas P can attack O-signed formulas repeatedly.

These asymmetries are introduced by dialogue conditions only. The argumentation forms themselves (as given in Definition 1.2) are symmetric with respect to the two players P and O. That is, they are independent of whether the assertion is made by the proponent P or by the opponent O; they are thus player independent.

Definition 1.7 *P wins a dialogue for a formula A* if the dialogue is finite, begins with the move $P\,A$ and ends with a move of P such that O cannot make another move. *winning a dialogue*

Remark 1.8 A dialogue won by P ends with a move $\langle \delta(n) = P\,a, \eta(n) = [m, Z] \rangle$, where a is an atomic formula.

Example 1.9 A dialogue for the formula $(a \lor b) \to \neg\neg(a \lor b)$ is the following:

0.	$P\,(a \lor b) \to \neg\neg(a \lor b)$	
1.	$O\,a \lor b$	$[0, A]$
2.	$P \lor$	$[1, A]$
3.	$O\,a$	$[2, D]$

(continued on next page)

4.	$P\,\neg\neg(a \vee b)$	$[1, D]$
5.	$O\,\neg(a \vee b)$	$[4, A]$
6.	$P\,a \vee b$	$[5, A]$
7.	$O\,\vee$	$[6, A]$
8.	$P\,a$	$[7, D]$

The dialogue starts with the assertion of the formula $(a \vee b) \to \neg\neg(a \vee b)$ by the proponent P in the initial move at position 0. This initial move is attacked ($\eta(1) = [0, A]$) by the opponent O with the assertion of the antecedent $a \vee b$ ($\delta(1) = O\,a \vee b$) of the implication asserted by P at position 0. The attack is thus made according to the argumentation form for implication.

At position 2, the proponent does not proceed according to the argumentation form for implication by defending O's attack move with the assertion of the succedent $\neg\neg(a \vee b)$ of the attacked implication. Instead, the proponent makes the symbolic attack $P\,\vee$ on O's assertion $a \vee b$. This move is thus made according to the argumentation form for disjunction. The attack is defended by O with the assertion of the left disjunct a (alternatively, O could also have chosen the right disjunct b). The moves at positions 1–3 are an instance of the argumentation form for disjunction.

As a is an atomic formula, it cannot be attacked. At position 4, the proponent defends O's attack $O\,a \vee b$ by asserting the succedent $\neg\neg(a \vee b)$ of the attacked implication $(a \vee b) \to \neg\neg(a \vee b)$. The moves at positions 0, 1 and 4 are an instance of the argumentation form for implication.

The opponent now attacks $P\,\neg\neg(a \vee b)$ at position 5 by asserting $O\,\neg(a \vee b)$ according to the argumentation form for negation. By this argumentation form there is no defense for the attack. But the proponent can attack $O\,\neg(a \vee b)$ with the assertion $P\,a \vee b$. The moves at positions 4 and 5 are an instance of the argumentation form for negation, and the moves at positions 5 and 6 are another instance of that argumentation form.

Next O attacks $P\,a \vee b$ with the symbolic attack $O\,\vee$ according to the argumentation form for disjunction at position 7. Finally, this attack is defended by P's assertion of the left disjunct a. The moves at positions 6–8 are made according to the argumentation form for disjunction. Note that P cannot defend here by asserting the right disjunct b: the opponent has not asserted the atomic formula b before, hence such a move is prohibited by condition ($D10$).

The proponent's move at position 8 is the last one. The opponent cannot attack a, since it is an atomic formula. Each other P-signed formula has been attacked by O, thus no more attack moves can be made by O due to condition ($D13$), as these would be repetitions of attacks already made. And since each proponent attack that can be defended according to an argumentation form has already been defended by O, no more defense moves are possible either, due to condition ($D12$). The dialogue is finite, begins with the move $P\,(a \vee b) \to \neg\neg(a \vee b)$ and ends with a move of P such that O cannot make another move; the dialogue for the formula $(a \vee b) \to \neg\neg(a \vee b)$ is thus won by P.

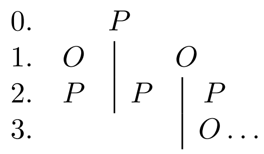
1.2.3 Strategies

We next introduce dialogue trees and define strategies. We explain first what we call a path.

Definition 1.8 A *path* in a branch of a tree with root node n_0 is a sequence $n_0, n_1, \ldots, n_k$ of nodes for $k \geq 0$ where n_i and n_{i+1} are adjacent for $0 \leq i < k$. *path*

Definition 1.9 A *dialogue tree* is a tree whose branches contain as paths all possible dialogues for a given formula. *dialogue tree*

Example 1.10 Schematic example of a dialogue tree:



At each odd position all possible moves for O have to be considered, and at each even position all possible moves for P have to be considered.

Remark 1.11 For a given formula A there is exactly one dialogue tree, if we consider trees to be equal modulo swapping of branches.

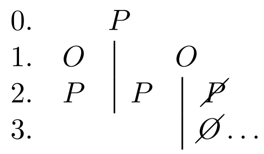
Definition 1.10 A *strategy* for a formula A is a subtree S of the dialogue tree for A such that *strategy*

(i) S does not branch at even positions,

(ii) S has as many nodes at odd positions as there are possible moves for O,

(iii) and all branches of S are dialogues for A won by P.

Example 1.12 Schematic example of a strategy:



At each odd position all possible moves for O have to be considered (ii), but at each even position only *one* move for P has to be considered (i). The two remaining branches are dialogues won by P (iii).

Remark 1.13 In more game-theoretic terms, the strategies defined here could also be called *winning strategies for the player P*, and a corresponding definition could be given of *winning strategies for the player O*. For the dialogical treatment of logic undertaken here, only the first notion is needed, however. We can thus simply speak of *strategies*.

Remark 1.14 Strategies are finite for propositional formulas. All the branches in a strategy have finite length by definition, whereas dialogues that are not part of a strategy can be of infinite length. Dialogue trees are therefore infinite objects in general. As dialogue trees can be constructed breadth-first, of course, an existing strategy can always be found.

Remark 1.15 Formulas can have no, exactly one or more than one strategy.

Example 1.16 There is exactly one strategy for the formula $a \to \neg\neg a$:

0.	$P\, a \to \neg\neg a$	
1.	$O\, a$	$[0, A]$
2.	$P\, \neg\neg a$	$[1, D]$
3.	$O\, \neg a$	$[2, A]$
4.	$P\, a$	$[1, A]$

The strategy contains only one branch.

Example 1.17 For the formula $(a \vee b) \to \neg\neg(a \vee b)$ there are the following three strategies, among others:

(i)

0.	$P\, (a \vee b) \to \neg\neg(a \vee b)$			
1.	$O\, a \vee b$			$[0, A]$
2.	$P\, \neg\neg(a \vee b)$			$[1, D]$
3.	$O\, \neg(a \vee b)$			$[2, A]$
4.	$P\, a \vee b$			$[3, A]$
5.	$O \vee$			$[4, A]$
6.	$P \vee$			$[1, A]$
7.	$O\, a$	$[6, D]$	$O\, b$	$[6, D]$
8.	$P\, a$	$[5, D]$	$P\, b$	$[5, D]$

(ii)

0.	$P\, (a \vee b) \to \neg\neg(a \vee b)$			
1.	$O\, a \vee b$			$[0, A]$
2.	$P\, \neg\neg(a \vee b)$			$[1, D]$
3.	$O\, \neg(a \vee b)$			$[2, A]$
4.	$P \vee$			$[1, A]$
5.	$O\, a$	$[4, D]$	$O\, b$	$[4, D]$
6.	$P\, a \vee b$	$[3, A]$	$P\, a \vee b$	$[3, A]$
7.	$O \vee$	$[6, A]$	$O \vee$	$[6, A]$
8.	$P\, a$	$[7, D]$	$P\, b$	$[7, D]$

(iii)

0.	$P\, (a \vee b) \to \neg\neg(a \vee b)$			
1.	$O\, a \vee b$			$[0, A]$
2.	$P \vee$			$[1, A]$
3.	$O\, a$	$[2, D]$	$O\, b$	$[2, D]$
4.	$P\, \neg\neg(a \vee b)$	$[1, D]$	$P\, \neg\neg(a \vee b)$	$[1, D]$
5.	$O\, \neg(a \vee b)$	$[4, A]$	$O\, \neg(a \vee b)$	$[4, A]$
6.	$P\, a \vee b$	$[5, A]$	$P\, a \vee b$	$[5, A]$
7.	$O \vee$	$[6, A]$	$O \vee$	$[6, A]$
8.	$P\, a$	$[7, D]$	$P\, b$	$[7, D]$

There are more strategies for this formula than the three shown here, because the proponent can repeatedly attack formulas asserted by the opponent. For example, in strategy (iii) the proponent could at position 4 (in the left as well as in the right dialogue) repeat the attack $P \vee$ on $O\, a \vee b$. The subtrees below these attacks (in both dialogues) would have the same form as the subtree below position 2 in strategy (iii).

Example 1.18 There is no strategy for the formula $a \vee \neg a$, an instance of *tertium non datur*. The only possible dialogue is

0.	$P\, a \vee \neg a$	
1.	$O \vee$	$[0, A]$
2.	$P\, \neg a$	$[1, D]$
3.	$O\, a$	$[2, A]$

and P does not win.

There would be a strategy, if condition ($D12$) were dropped for P. Then P could defend the attack $O \vee$ a second time by stating a, thereby winning the dialogue. Condition ($D11$) does not have to be dropped because there are not more than one open attacks at position 3 (there is exactly one open attack at position 3; the attack $O \vee$ is not open there since it has already been defended at position 2).

1.2.4 Completeness

Definition 1.11 A formula A is called *dialogue-provable* (or DI^p*-dialogue-provable*) if there is a strategy for A. Notation: $\vdash_{DI^p} A$. *dialogue-provable*

Remark 1.19 We speak of dialogue-*provable* formulas here, in accordance with Felscher [2002]. Contrasting Gentzen's calculi with dialogues, Felscher [2002, p. 127] remarks:

> Gentzen's calculi of proofs are easily explained in that they represent the weakest consequence relation for which the provability interpretation is valid. The connection between dialogues and the argumentative interpretation of logical operations is [...] located on a different level: it is not the dialogues but the *strategies* for dialogues which will correspond to proofs. I thus formulate the *basic purpose* for the use of dialogues:
>
> (A_0) Logically provable assertions shall be those which, for *purely formal* reasons, can be upheld by a strategy covering every dialogue chosen by [O].

However, the fact that we speak of *provability* in the context of dialogues (thus following Felscher) should not be misunderstood in a way that would imply that dialogues cannot be seen as a (formal) semantics (as opposed to considering dialogues only as a proof system or calculus).

Of course, such a misunderstanding could only arise if one's notion of semantics is limited to truth-conditional semantics, as opposed to proof-theoretic semantics (like Brouwer–Heyting–Kolmogorov (BHK) semantics, or related justificationist, verificationist, pragmatist or falsificationist approaches in the tradition of Dummett and Prawitz) where the notion of proof or closely related notions are of central importance.

As the meaning of the logical constants is in some sense given by the argumentation forms in terms of how assertions containing the logical constants can be used in an argumentation, dialogues might very well be seen as a semantics under the heading "meaning is use", and were indeed introduced for that purpose. This aspect can be emphasized by speaking of *(logical) validity* instead of dialogue-provability. *validity*

Theorem 1.20 (Completeness) *The dialogue-provable formulas are exactly the formulas provable in intuitionistic logic.*

Remark 1.21 This theorem has been shown (also for intuitionistic first-order logic) by Felscher [1985] by proving for Gentzen's sequent calculus *LJ* (for intuitionistic first-order logic; see Gentzen [1935]) that every (first-order) strategy can be transformed into a proof in *LJ*, and vice versa.

1.3 Addendum: Contraction in dialogues

In dialogues, the structural operations of thinning and contraction are only implicitly given by the dialogue conditions. This is comparable to natural deduction, where these structural operations are also only implicitly given, namely by how assumptions are discharged. Whereas in sequent calculus these operations are explicitly given as structural rules. That the structural operations are only implicitly given in dialogues can be seen as an advantage: we have argumentation forms only for the logical constants, and everything else is—in part implicitly—taken care of by the dialogue conditions.

We now consider contraction, which will be of importance at the end of the second lecture.

Remark 1.22 In dialogues, the twofold use made by the proponent P of a formula A asserted by the opponent O corresponds to the structural operation of contraction, contracting A, A into A. The twofold use can consist either

(1) in the twofold attack of a formula by the proponent P,

(2) in the twofold assertion by the proponent P of a formula asserted by the opponent O before,

or

(3) in an attack of a formula A by the proponent P together with the assertion of A by P.

That is, the twofold use can be of the following forms:

(1)
$$\begin{array}{lll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,e & [k, A] \\ & \vdots & \\ m. & P\,e & [k, A] \end{array}$$

(2)
$$\begin{array}{lll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,A & [i < l, Z] \\ & \vdots & \\ m. & P\,A & [j < m, Z] \end{array}$$

(3)
$$\begin{array}{lll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,e & [k, A] \\ & \vdots & \\ m. & P\,A & [i < m, Z] \end{array} \quad \text{respectively} \quad \begin{array}{lll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,A & [i < l, Z] \\ & \vdots & \\ m. & P\,e & [k, A] \end{array}$$

Example 1.23 In the following two examples the twofold use made by P of an assertion made by O is of the form (1). The formulas $\neg(a \wedge \neg a)$ respectively $\neg\neg(a \vee \neg a)$ are not provable without a twofold attack on $a \wedge \neg a$ respectively $\neg(a \vee \neg a)$ by P, or without the corresponding discharge of two occurrences of the same assumption in the natural deduction derivations (where $\neg a := a{\to}\bot$), respectively:

(i)

0.	$P\,\neg(a \wedge \neg a)$	
1.	$O\,a \wedge \neg a$	$[0, A]$
2.	$P\,\wedge_1$	$[1, A]$
3.	$O\,a$	$[2, D]$
4.	$P\,\wedge_2$	$[1, A]$
5.	$O\,\neg a$	$[4, D]$
6.	$P\,a$	$[5, A]$

$$\dfrac{\dfrac{\dfrac{[a \wedge \neg a]^1}{a}\,(\wedge\,\mathrm{E}) \qquad \dfrac{[a \wedge \neg a]^1}{\neg a}\,(\wedge\,\mathrm{E})}{\bot}\,(\to\mathrm{E})}{\neg(a \wedge \neg a)}\,(\to\mathrm{I})^1$$

The twofold attack at positions 2 and 4 corresponds to the contraction of $a \wedge \neg a, a \wedge \neg a$ to $a \wedge \neg a$.

(ii)

0.	$P\,\neg\neg(a \vee \neg a)$	
1.	$O\,\neg(a \vee \neg a)$	$[0, A]$
2.	$P\,a \vee \neg a$	$[1, A]$
3.	$O\,\vee$	$[2, A]$
4.	$P\,\neg a$	$[3, D]$
5.	$O\,a$	$[4, A]$
6.	$P\,a \vee \neg a$	$[1, A]$
7.	$O\,\vee$	$[6, A]$
8.	$P\,a$	$[7, D]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a]^1}{a \vee \neg a}\,(\vee\,\mathrm{I}) \qquad [\neg(a \vee \neg a)]^2}{\bot}\,(\to\mathrm{E})}{\neg a}\,(\to\mathrm{I})^1}{a \vee \neg a}\,(\vee\,\mathrm{I}) \qquad [\neg(a \vee \neg a)]^2}{\bot}\,(\to\mathrm{E})}{\neg\neg(a \vee \neg a)}\,(\to\mathrm{I})^2$$

The twofold attack at positions 2 and 6 corresponds to the contraction of $\neg(a \vee \neg a), \neg(a \vee \neg a)$ to $\neg(a \vee \neg a)$.

1.4 Addendum: Classical dialogues

Although we are only concerned with intuitionistic logic, we point out here how dialogues for classical (propositional) logic relate to dialogues for intuitionistic (propositional) logic.

Remark 1.24 If the conditions ($D11$) and ($D12$) are restricted to apply only to O (and no more to P), then the formulas provable on the basis of the thus modified dialogues are exactly the formulas provable in classical logic.

Definition 1.12 A *classical dialogue* is a dialogue where the conditions ($D11$) and ($D12$) do hold for O but not for P, that is, where conditions ($D11$) and ($D12$) are replaced by the following conditions ($D11^+$) and ($D12^+$), respectively:

classical dialogue

($D11^+$) If $\eta(p) = [n, D]$ for even n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.

That is, if at a position $p - 1$ there are more than one open attacks by P, then only the last of them may be defended by O at position p.

($D12^+$) For every even m there is at most one n such that $\eta(n) = [m, D]$.

That is, an attack by P may be defended by O at most once.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for dialogues are directly carried over to the corresponding notions for classical dialogues.

Example 1.25 There is a classical strategy for the formula $a \vee \neg a$:

0.	$P\, a \vee \neg a$	
1.	$O \vee$	$[0, A]$
2.	$P\, \neg a$	$[1, D]$
3.	$O\, a$	$[2, A]$
4.	$P\, a$	$[1, D]$

The last move is possible due to the replacement of condition ($D12$) by condition ($D12^{+}$). In the presence of ($D12$) this move is not possible, and there is thus no DI^p-strategy for (any instance of) *tertium non datur* (cf. Example 1.18).

Example 1.26 There is a classical strategy for the formula $\neg\neg a \to a$:

0.	$P\, \neg\neg a \to a$	
1.	$O\, \neg\neg a$	$[0, A]$
2.	$P\, \neg a$	$[1, A]$
3.	$O\, a$	$[2, A]$
4.	$P\, a$	$[1, D]$

The last move is possible due to the replacement of condition ($D11$) by condition ($D11^{+}$). In the presence of ($D11$) this move is not possible, and there is thus no DI^p-strategy for (any instance of) double negation elimination.

In the following we will not consider classical dialogues again. We consider only intuitionistic logic.

Second Lecture: Dialogues for Definitional Reasoning

2.1 Summary of the first lecture

In the last lecture we explained the meanings of the logical constants $\wedge, \vee, \to$ and $\neg$ in certain game-theoretical terms. For each logical constant we gave an *argumentation form* which determines how an assertion (having the logical constant in outermost position) made by one player X can be attacked by the other player Y and how this attack can be defended by X. For example, the argumentation form for '$\to$' is:

assertion:	$X\ A \to B$
attack:	$Y\ A$
defense:	$X\ B$

This gives an *argumentative interpretation* of implication '$\to$' in the following sense: An argument on an implicative assertion $A \to B$ is reduced to an argument on B under the assumption A.

In addition to the argumentation forms we gave certain conditions on how exactly the two players can make their *moves*: The proponent P makes the first move (asserting a complex formula), and the two players proponent P and opponent O make moves alternatingly as determined by the argumentation forms (see conditions ($D00$), ($D01$) and ($D02$)). Furthermore

($D10$) P may assert an atomic formula only if it has been asserted by O before,

($D11$) if at a position $p-1$ there are more than one open attacks, then only the last of them may be defended at position p.

($D12$) an attack may be defended at most once,

($D13$) and a P-signed formula may be attacked at most once.

These *dialogue conditions* (together with the argumentation forms) define the notion of *dialogue*.

We then said what it means that P *wins a dialogue for a formula* A, namely that the dialogue is finite, begins with the move PA and ends with a move by P such that O cannot make another move.

There is a winning strategy for P (short: *strategy*) if and only if for each possible move by the opponent O the proponent P can make another move such that in the end each dialogue for the given formula is won by P. Whereas dialogues are just linear sequences of moves, strategies are in general trees which branch at odd positions, that is, at O-moves. We saw, for example, that there is a strategy for the formula $(a \vee b) \to \neg\neg(a \vee b)$:

0.	$P\ (a \vee b) \to \neg\neg(a \vee b)$	
1.	$O\ a \vee b$	$[0, A]$

(*continued on next page*)

2.		$P \vee$		$[1, A]$
3.	$O\, a$	$[2, D]$	$O\, b$	$[2, D]$
4.	$P \neg\neg(a \vee b)$	$[1, D]$	$P \neg\neg(a \vee b)$	$[1, D]$
5.	$O \neg(a \vee b)$	$[4, A]$	$O \neg(a \vee b)$	$[4, A]$
6.	$P\, a \vee b$	$[5, A]$	$P\, a \vee b$	$[5, A]$
7.	$O \vee$	$[6, A]$	$O \vee$	$[6, A]$
8.	$P\, a$	$[7, D]$	$P\, b$	$[7, D]$

There is no strategy for the formula $a \vee \neg a$, an instance of *tertium non datur*. The only possible dialogue is

0.	$P\, a \vee \neg a$	
1.	$O \vee$	$[0, A]$
2.	$P \neg a$	$[1, D]$
3.	$O\, a$	$[2, A]$

and P does not win.

Strategies give us a notion of *logical validity* for intuitionistic logic, and we have the following completeness result: There is a strategy for a formula A if and only if A is provable in intuitionistic logic.

Concerning the dialogues considered so far, we can observe the following:

(i) It is not possible to attack assertions of *atomic* formulas $(a, b, c, \ldots)$.

(ii) Dialogues won by P always end with the assertion of an *atomic* formula.

Compare the two following dialogues:

0.	$P\, (a \vee b) \to \neg\neg(a \vee b)$		0.	$P\, (a \vee b) \to \neg\neg(a \vee b)$	
1.	$O\, a \vee b$	$[0, A]$	1.	$O\, a \vee b$	$[0, A]$
2.	$P \vee$	$[1, A]$		|	
3.	$O\, a$	$[2, D]$		|	
4.	$P \neg\neg(a \vee b)$	$[1, D]$	2.	$P \neg\neg(a \vee b)$	$[1, D]$
5.	$O \neg(a \vee b)$	$[4, A]$	3.	$O \neg(a \vee b)$	$[2, A]$
6.	$P\, a \vee b$	$[5, A]$	4.	$P\, a \vee b$	$[3, A]$
7.	$O \vee$	$[6, A]$			
8.	$P\, a$	$[7, D]$			

The first dialogue is won by P. The second dialogue is *not* won by P, since O can attack the assertion $a \vee b$ made by P in the last move at position 4 with the move $\langle \delta(5) = O \vee, \eta(5) = [4, A] \rangle$.

2.2 Introduction

In this second lecture we will consider extensions of logic by a certain kind of definitions for atoms, where the defining conditions are not restricted to atomic formulas but can be given by arbitrary (first-order) formulas. These definitions are thus a generalization of monotone inductive definitions or, equivalently, of (implication-free) definite Horn clause programs as they are used in standard logic programming based on the resolution principle.

We introduce dialogues containing the principles of definitional reflection and definitional closure as an additional argumentation form of definitional reasoning.

The clausal definitions need not be wellfounded. This leads to paradoxes like Russell's, whose dialogical treatment will be considered as an example of definitional reasoning. The example shows that the structural operation of contraction can be critical in the presence of non-wellfounded clausal definitions: without further restrictions, there are then strategies for contradictory assertions.

Definitional dialogues will be introduced in two steps:

(1) As we want to reason about definitions whose defining conditions can be complex formulas, we have to make sure that it is possible that dialogues in a strategy can not only end with P-moves asserting atomic formulas, but that they can also end with P-moves asserting complex formulas.

We first introduce so-called EI^p_c-dialogues with this property. For this kind of dialogues there is also a completeness result with respect to intuitionistic logic.

(2) We then introduce an argumentation form for definitional reasoning, and define definitional dialogues on the basis of EI^p_c-dialogues.

2.3 EI^p- and EI^p_c-dialogues

We first define EI^p-dialogues as a restricted form of DI^p-dialogues (which have been introduced in the first lecture). They differ from DI^p-dialogues only in that each opponent move must now refer to the immediately preceding proponent move. This restriction yields certain technical advantages, without changing the set of dialogue-provable formulas extensionally.

Definition 2.1 An *EI^p-dialogue* is a DI^p-dialogue with the additional condition *EI^p-dialogue*

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$.

That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n-1$.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for DI^p-dialogues are directly carried over to the corresponding notions for EI^p-dialogues.

Remark 2.1 The EI^p-dialogues as they are defined here are exactly the E-dialogues of Felscher [1985, 2002] (references to their original formulation are given therein).

Definition 2.2 A formula A is called *EI^p-dialogue-provable* if there is an EI^p-strategy for A. Notation: $\vdash_{EI^p} A$. *EI^p-dialogue-provable*

Remark 2.2 It has been shown by Felscher that there is a recursive algorithm by which every EI^p-strategy can be embedded into a DI^p-strategy, and that therefore the EI^p-dialogue-provable formulas are exactly the formulas provable in intuitionistic propositional logic (see Felscher [1985, p. 221] and Felscher

[2002, p. 119]; these results hold not only for the propositional but also for the first-order case). As the DI^p-dialogue-provable formulas are also exactly the formulas provable in intuitionistic propositional logic, the following holds: $\vdash_{EI^p} A$ *if and only if* $\vdash_{DI^p} A$.

Now we can define EI^p_c-dialogues as follows:

Definition 2.3 An EI^p_c*-dialogue* is an EI^p-dialogue with the additional condition

EI^p_c-dialogue

($D14$) O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

Again, the notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for DI^p-dialogues are directly carried over to the corresponding notions for EI^p_c-dialogues.

Remark 2.3 Condition (E) implies condition ($D13$). Furthermore, condition (E) implies condition ($D11$) for odd p and condition ($D12$) for odd n (cf. Definition 1.6).

In the presence of condition (E), condition ($D13$) can therefore be omitted, and conditions ($D11$) and ($D12$) can be restricted to conditions ($D11'$) and ($D12'$), respectively, as follows:

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.

That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$.

That is, an attack by O may be defended by P at most once.

Example 2.4 EI^p_c-dialogues won by P need not end with the assertion of an *atomic* formula, but can end with the assertion of a *complex* formula.

Consider the following EI^p_c-dialogue for $(a \vee b) \rightarrow \neg\neg(a \vee b)$:

0.	$P\ (a \vee b) \rightarrow \neg\neg(a \vee b)$	
1.	$O\ a \vee b$	$[0, A]$
2.	$P\ \neg\neg(a \vee b)$	$[1, D]$
3.	$O\ \neg(a \vee b)$	$[2, A]$
4.	$P\ a \vee b$	$[3, A]$
~~5.~~	~~$O\ \vee$~~	~~$[4, A]$~~

The dialogue is won by P, and it is a winning strategy for $(a \vee b) \rightarrow \neg\neg(a \vee b)$.

The opponent O cannot attack the assertion $a \vee b$ made by P in the last move at position 4 with the move $\langle \delta(5) = O\,\vee, \eta(5) = [4, A] \rangle$ anymore, due to condition ($D14$): the formula $a \vee b$ has already been asserted by O at position 1, without having been attacked by P.

Definition 2.4 A formula A is called EI^p_c*-dialogue-provable* if there is an EI^p_c-strategy for A. Notation: $\vdash_{EI^p_c} A$.

EI^p_c-dialogue-provable

Theorem 2.5 (Completeness) *The EI_c^p-dialogue-provable formulas are exactly the formulas provable in intuitionistic logic.*

Remark 2.6 Completeness has been proved constructively by showing that there is an EI_c^p-strategy for a formula A if and only if A is provable in sequent calculus with complex initial sequents $B \vdash B$ (where B is complex or atomic) for intuitionistic logic.

This result is the theoretical basis for the introduction of definitional dialogues, which will allow us to reason about definitions whose defining conditions can be complex formulas.

2.4 Clausal definitions

We introduce the argumentation form of definitional reasoning for clausal definitions. Clausal definitions are collections of definitional clauses, which are formulated over a first-order language. We restrict ourselves to the quantifier-free fragment.

Definition 2.5 We extend our language to a (quantifier-free) *first-order language*, where for *variables* $x, y, \ldots$, *(individual) constants* $k, l, m, \ldots$ and *function symbols* $f, g, \ldots$ we define *terms* as follows: *first-order language*

(i) Every variable is a term. *terms*

(ii) Every individual constant is a term.

(iii) If f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is also a term.

We now use $a, b, c, \ldots$ also as *relation symbols* (or *predicate symbols*). If a is an n-ary relation symbol and if $t_1, \ldots, t_n$ are terms, then $a(t_1, \ldots, t_n)$ is an *atomic formula* (*atom*). Complex formulas are defined as usual. *relation symbols*

Definition 2.6 A *definitional clause* is an expression of the form *definitional clause*

$$a \Leftarrow B_1 \wedge \ldots \wedge B_n$$

for $n \geq 0$, where a is atomic and the B_i in the *body* $B_1 \wedge \ldots \wedge B_n$ of the clause are the *defining conditions* for the *head* a. (The symbol '$\Leftarrow$' is used exclusively to write definitional clauses and should not be confused with implication '$\rightarrow$'.) The defining conditions B_i need not be atomic but can be any complex formula. Clauses with empty body are called *facts*; we indicate empty bodies with the symbol '$\top$' (*verum*). *defining conditions*

Example 2.7 (i) $a \Leftarrow (b \rightarrow c) \wedge d$ is a (propositional) definitional clause with head a and body $(b \rightarrow c) \wedge d$, containing the two defining conditions $b \rightarrow c$ and d. (This clause can also be read as a first-order clause in which all relation symbols have arity 0.)

(ii) $a(x, y) \Leftarrow \neg b(k, l, x)$ is a (quantifier-free) first-order definitional clause with the binary relation $a(x, y)$ in the head and having as defining condition the complex formula $\neg b(k, l, x)$.

Definition 2.7 A finite set $\mathcal{D}$ of definitional clauses *definition of atom*

$$\mathcal{D}\begin{cases} a \Leftarrow \Gamma_1 \\ \vdots \\ a \Leftarrow \Gamma_k \end{cases}$$

is a *(clausal) definition of the atom* a, where $\Gamma_i = B^i_1 \wedge \ldots \wedge B^i_{n_i}$ is the body of the i-th clause (for $1 \leq i \leq k$). These clauses are the *defining clauses* of a with respect to definition $\mathcal{D}$. *defining clauses*

The *set of defining conditions of* a will be represented by $\mathcal{D}(a)$, that is, $\mathcal{D}(a) = \{\Gamma_1, \ldots, \Gamma_k\}$.

Remark 2.8 We write the bodies Γ_i of definitional clauses as conjunctions

$$B^i_1 \wedge \ldots \wedge B^i_{n_i}$$

of the defining conditions $B^i_{l_i}$.

They could also be written as a list or set $B^i_1, \ldots, B^i_{n_i}$, where the comma functions as a 'structural conjunction'. The latter notation is more convenient in a sequent calculus setting. However, for dialogues we would first have to introduce a means to handle such lists or sets, whereas we can handle conjunctions directly via the argumentation form for $\wedge$. We will therefore use the former notation throughout.

Definition 2.8 A *definition* is any finite set of definitional clauses. Definitions $\mathcal{D}$ have thus the general form *definition*

$$\mathcal{D}\begin{cases} a_1 \Leftarrow \Gamma^1_1 \\ \vdots \\ a_1 \Leftarrow \Gamma^1_{k_1} \\ \vdots \\ a_n \Leftarrow \Gamma^n_1 \\ \vdots \\ a_n \Leftarrow \Gamma^n_{k_n} \end{cases}$$

(In logic programming terms, definitions $\mathcal{D}$ are (a generalization of) logic programs where the bodies of program clauses can be arbitrary formulas.)

2.5 Definitional reasoning

We can now define an argumentation form that will allow us to reason about such definitions.

Definition 2.9 For each atom a defined by definitional clauses

$$a \Leftarrow B^i_1 \wedge \ldots \wedge B^i_{n_i}$$

with defining conditions

$$\Gamma_i = B^i_1 \wedge \ldots \wedge B^i_{n_i} \quad \text{(where } 1 \leq i \leq k\text{)}$$

the following argumentation form of *definitional reasoning* determines how an atom a that is stated by X can be attacked by Y and how this attack can be defended by X. We use '$\mathscr{D}$' as a special symbol to indicate the attack. *definitional reasoning*

definitional reasoning: assertion: $X\, a$
attack: $Y\, \mathscr{D}$ (only if $a \neq \top$)
defense: $X\, \Gamma_i$ (X chooses $i = 1, \ldots, k$)

For the *verum* $\top$ we impose the following restriction: The move $X\, \top$ cannot be attacked with $Y\, \mathscr{D}$.

Remark 2.9 We have defined the argumentation form of definitional reasoning in such a way that atoms—with the exception of the *verum* $\top$—can be attacked independently of whether there are definitional clauses having these atoms in their head or not. In other words, whenever a player asserts an atom, the other player may ask for its definition, regardless of whether one has been given or not. And we will not give any dialogue conditions which would prohibit attacks on undefined atoms just because they are undefined.

The restriction with respect to the *verum* $\top$ is necessary if $\top$ is treated as an atomic formula. Otherwise it would be attackable as well. This would be in conflict with its intended meaning, suggested by its use as an indicator of empty bodies of definitional clauses, that is, by standing for the empty conjunction. The meaning of the *verum* $\top$ is stipulated by the imposed restriction.

Remark 2.10 The argumentation form of definitional reasoning is formulated for atoms a defined by definitional clauses

$$\begin{array}{c} a \Leftarrow B^1_1 \wedge \ldots \wedge B^1_{n_1} \\ \vdots \\ a \Leftarrow B^k_1 \wedge \ldots \wedge B^k_{n_k} \end{array}$$

That is, in definitional reasoning the Γ_i chosen by X in a defense to an attack $Y\, \mathscr{D}$ on $X\, a$ must be the body of a clause with head a in the case of propositional clauses; bodies of definitional clauses not defining a cannot be chosen.

Remark 2.11 The argumentation form of definitional reasoning comprises the two principles of definitional reflection and definitional closure, which have been introduced as sequent-style inferences by Hallnäs and Schroeder-Heister [1990, 1991] (see also Hallnäs [1991] and Schroeder-Heister [1993]).

In natural deduction, these principles can be formulated as follows. Let the atom a be defined by

$$\mathcal{D} \begin{cases} a \Leftarrow \Gamma_1 \\ \quad \vdots \\ a \Leftarrow \Gamma_k \end{cases}$$

Then, for $1 \leq i \leq k$, the *principle of definitional closure* takes the form of an introduction rule for atoms a: *definitional closure*

$$\frac{\Gamma_i}{a}\ (\text{def. closure})$$

And the *principle of definitional reflection* takes the form of a (general) elimination rule for atoms a: *definitional reflection*

$$\dfrac{a \quad \begin{matrix}[\Gamma_1] \\ \vdots \\ C\end{matrix} \quad \cdots \quad \begin{matrix}[\Gamma_k] \\ \vdots \\ C\end{matrix}}{C}\ \text{(def. reflection)}$$

The principle of definitional reflection is related to the *inversion principle* (see Prawitz [1965]) and can be expressed as follows: *inversion principle*

> Whatever formula C follows from each of the defining conditions $\Gamma_1, \dots, \Gamma_k$ of the atom a follows from a itself.

The principle of definitional reflection is justified if given definitions of atoms can be assumed to be complete in the sense that the atoms are defined by the given definitional clauses *and by nothing else*. In mathematical definitions this is sometimes made explicit by giving definitional clauses for something together with the remark that *nothing else* defines that something, or by saying that one defines *the smallest set* for which some given definitional clauses hold.

Remark 2.12 The argumentation form of definitional reasoning is the dialogical equivalent to the principles of definitional closure and definitional reflection. Both principles are incorporated in the one argumentation form of definitional reasoning.

For dialogues, the difference between definitional closure and definitional reflection appears on the level of strategies. Here only one defense move $P\,\Gamma_i$ has to be given for an attack $O\,\mathscr{D}$, whereas all possible defense moves $O\,\Gamma_i$ have to be given for an attack $P\,\mathscr{D}$. In other words, in the first case only the defining conditions Γ_i of one clause defining the attacked atom have to be given, whereas in the second case the defining conditions Γ_i of each clause defining the attacked atom have to be given.

Thus definitional reasoning in dialogues corresponds to the principles of definitional closure and definitional reflection in natural deduction as follows:

(i) Instances of the argumentation form of definitional reasoning in which the attack move is $O\,\mathscr{D}$ correspond to applications of definitional closure, and

(ii) instances of the argumentation form of definitional reasoning in which the attack move is $P\,\mathscr{D}$ correspond to applications of definitional reflection.

2.6 Definitional dialogues

Next we will introduce definitional dialogues based on EI_c^p-dialogues.

Definition 2.10 *Definitional dialogues* are EI_c^p-dialogues where the following changes are made: *definitional dialogues*

The conditions ($D00$) and ($D01$) are replaced by the following conditions ($D00'$) and ($D01'$), where the restriction of the expressions in $\delta(0)$ and $\delta(m)$ to complex formulas is discarded; that is, a definitional dialogue can start with the assertion of an atomic formula, and atomic formulas can be attacked:

($D00'$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01'$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

Condition ($D02$) remains without change.

Condition ($D10$) is omitted altogether, so that P can now assert atomic formulas without O having asserted them before. Conditions ($D11'$), ($D12'$) and (E) remain without change. Condition ($D14$) is replaced by the following condition ($D14^*$) which is ($D14$) restricted to complex formulas:

($D14^*$) O can attack a complex formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

The following condition is added in order to prohibit attacks by O on atoms asserted by O before:

($D15$) If for an atom a there is a move $\langle \delta(l) = O\, a, \eta(l) = [k, Z] \rangle$, then there is no attack $\langle \delta(n) = O\, \mathscr{D}, \eta(n) = [m, A] \rangle$ for $\delta(m) = P\, a$ with $k < l < m < n$.

That is, O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

Furthermore, the following proviso for applications of definitional reasoning in the presence of variables is added (where we write $A\sigma$ to denote the result of the application of a substitution σ to a formula A):

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack $P\, \mathscr{D}$ on a is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for EI_c^p-dialogues are directly carried over to the corresponding notions for definitional dialogues.

Remark 2.13 The omission of condition ($D10$) is compensated by the fact that O can attack any atom asserted by P with a move $O\, \mathscr{D}$.

The restriction of condition ($D14$) to complex formulas (yielding condition ($D14^*$)) was not necessary in the treatment of EI_c^p-dialogues because attacks on atomic formulas are not possible there.

2.6.1 Examples for propositional definitional reasoning

Example 2.14 We consider the definition

$$\mathcal{D}_1 \begin{cases} a \Leftarrow \top \\ d \Leftarrow \top \\ d \Leftarrow a \\ c \Leftarrow a \wedge d \end{cases}$$

With respect to $\mathcal{D}_1$, the following is a strategy for the atom c:

0.	$P\,c$			
1.	$O\,\mathscr{D}$	$[0, A]$		
2.	$P\,a \wedge d$	$[1, D]$		
3.	$O\,\wedge_1$	$[2, A]$	$O\,\wedge_2$	$[2, A]$
4.	$P\,a$	$[3, D]$	$P\,d$	$[3, D]$
5.	$O\,\mathscr{D}$	$[4, A]$	$O\,\mathscr{D}$	$[4, A]$
6.	$P\,\top$	$[5, D]$	$P\,a$	$[5, D]$
7.			$O\,\mathscr{D}$	$[6, A]$
8.			$P\,\top$	$[7, D]$

At position 0 the proponent P asserts the atom c. In definitional dialogues this is allowed by condition ($D00'$), whereas in standard dialogues with condition ($D00$) only complex formulas can be asserted in initial moves at position 0. At position 1 this assertion is attacked by O according to the argumentation form of definitional reasoning. The proponent P defends this attack by asserting the defining conditions $a \wedge d$ of the attacked atom c, as given by the last clause of definition $\mathcal{D}_1$. The opponent O attacks $a \wedge d$ at position 3, and P defends at position 4 by asserting the atoms a and d, respectively. The proponent P can assert the atomic formulas a and d—without O having asserted them before—as there is no condition ($D10$) in definitional dialogues, which would prohibit these moves. However, the opponent O can attack any atoms asserted by P (if not prohibited by condition ($D15$)), and does so with the move $O\,\mathscr{D}$ at position 5 in each of the two dialogues.

In the left dialogue, the proponent defends the opponent's attack on a by asserting $\top$ at position 6 (there are no defining conditions for the atom a; it is given as a fact by the first clause in $\mathcal{D}_1$). In the right dialogue, the proponent chooses to defend by asserting the defining condition a of d, as given in the third clause of $\mathcal{D}_1$. The right dialogue then proceeds as the left one. Alternatively, the proponent could have defended the opponent's attack by choosing to use the second clause of $\mathcal{D}_1$. This clause gives d as a fact, and the proponent's defense would thus be the *verum* $\top$. That is, the right dialogue would end with the move $P\,\top$ already at position 6.

Both dialogues in the above strategy end with the assertion of the *verum* $\top$. As there is no attack possible on $\top$, both dialogues are won by P. The strategy contains only such applications of definitional reasoning in which the opponent attacks atomic formulas with moves $O\,\mathscr{D}$; that is, only the principle of definitional closure is employed here.

Example 2.15 An example where the principle of definitional reflection is used with respect to the definition $\mathcal{D}_1$ (just given in Example 2.14) is the following strategy for the formula $d \to a$:

0.	$P\,d \to a$			
1.	$O\,d$	$[0, A]$		
2.	$P\,\mathscr{D}$	$[1, A]$		
3.	$O\,\top$	$[2, D]$	$O\,a$	$[2, D]$
4.	$P\,a$	$[1, D]$	$P\,a$	$[1, D]$
5.	$O\,\mathscr{D}$	$[4, A]$		
6.	$P\,\top$	$[5, D]$		

The first application of definitional reasoning (comprising positions 1–3) is according to the principle of definitional reflection. Here the defining conditions of each of the definitional clauses for the attacked atom d have to be considered. As $\mathcal{D}_1$ contains two clauses for d, there are two defense moves (made at position 3) to be considered. In the left dialogue, the proponent can only defend the opponent's attack made at position 1 by asserting the atom a. The following attack by O, asking for defining conditions of a, is defended by P with $\top$ (using the first clause of $\mathcal{D}_1$, which is the only definitional clause for a). Here the principle of definitional closure has been employed. In the right dialogue, the proponent makes the same defense move at position 4 as in the left dialogue. Due to condition ($D15$) the opponent cannot attack the atom a: O has asserted a before (at position 3).

The proponent could also make the move $P\,\mathscr{D}$ at position 4 in the right dialogue instead. The dialogue would then end thus:

$$\begin{array}{lll} & \vdots & \\ 3. & O\,a & [2, D] \\ 4. & P\,\mathscr{D} & [3, A] \\ 5. & O\,\top & [4, D] \\ 6. & P\,a & [1, D] \end{array}$$

This yields a strategy in which the principle of definitional reflection has been employed twice.

2.6.2 Examples for first-order definitional reasoning

Definition 2.11 A substitution σ is a *unifier* of two atoms a and b if $a\sigma \equiv b\sigma$, that is, if $a\sigma$ and $b\sigma$ are syntactically identical. *unifier*

A substitution σ is a *most general unifier* of two atoms a and b if for all unifiers τ of a and b it holds that $\tau = \sigma\rho$ for a substitution ρ. *most general unifier*

Remark 2.16 In the case of first-order clauses one has to consider substitution instances of heads and bodies of clauses.

Let the substitution σ be a most general unifier for the atom a and the head a' of at least one first-order clause. Then the body Γ_i of such a clause with head a' can be chosen in a defense $X\,\Gamma_i\sigma$ to an attack $Y\,\mathscr{D}$ on $X\,a$ since $a\sigma \equiv a'\sigma$. That is, in order to defend such an attack, we first have to look for a most general unifier σ which unifies a with the head of a clause $a' \Leftarrow \Gamma_i$. If it exists (this is decidable by the unification algorithm), we apply it to Γ_i, and the defense move is $X\,\Gamma_i\sigma$.

For example, if the first-order clause $a(t) \Leftarrow b(x)$ is given by definition, then an attack $Y\,\mathscr{D}$ on a move $X\,a(x)$ can be defended with the move $b(t)$. That is, the definitional reasoning for the given clause is of the form

$$\begin{array}{l} X\,a(x) \\ Y\,\mathscr{D} \\ X\,b(t) \end{array}$$

where the substitution $\sigma = [t/x]$ is here the most general unifier for the atom $a(x)$ and the head $a(t)$ of the definitional clause. Applying σ to the body $b(x)$ of the clause yields $b(t)$, which is asserted in the defense move.

Example 2.17 We now consider the following (first-order) definition $\mathcal{D}_2$, in which the atoms $even(x)$ and $odd(x)$ are two unary relation symbols, and s is a unary function symbol (interpreted as the successor function on natural numbers):

$$\mathcal{D}_2 \begin{cases} even(0) \Leftarrow \top \\ even(s(x)) \Leftarrow odd(x) \\ odd(x) \Leftarrow \neg\, even(x) \end{cases}$$

Then for the given definition $\mathcal{D}_2$ the following definitional dialogue is a strategy for $\neg\, even(s(0))$:

0.	$P \neg\, even(s(0))$	
1.	$O\ even(s(0))$	$[0, A]$
2.	$P\,\mathscr{D}$ (variable binding: $[0/x]$)	$[1, A]$
3.	$O\ odd(0)$	$[2, D]$
4.	$P\,\mathscr{D}$ (variable binding: $[0/x]$)	$[3, A]$
5.	$O \neg\, even(0)$	$[4, D]$
6.	$P\ even(0)$	$[5, A]$
7.	$O\,\mathscr{D}$	$[6, A]$
8.	$P\,\top$	$[7, D]$

The applications of definitional reasoning comprising the moves at positions 1–3 and 3–5, respectively, are according to the principle of definitional reflection. The opponent's first defense move depends on the substitution $[0/x]$, which unifies the attacked atom $even(s(0))$ with the head $even(s(x))$ of clause 2 and yields the corresponding defining condition $odd(x)[0/x] = odd(0)$, asserted by O at position 3. The opponent's second defense move depends on the same substitution $[0/x]$; it unifies $odd(0)$ with the head $odd(x)$ of the third clause, allowing the opponent to defend with the defining condition $\neg\, even(x)[0/x] = \neg\, even(0)$ in the move at position 5. The moves at positions 6–8 are definitional reasoning by the principle of definitional closure. As $\top$ cannot be attacked, the dialogue ends with the proponent's move at position 8. By reasoning about the definition $\mathcal{D}_2$ we have thus shown $\neg\, even(s(0))$.

From a logic programming perspective this can be described as follows: The initial move expresses in a formal way a query about the given definition (or program) $\mathcal{D}_2$ like "Does $\neg\, even(s(0))$ hold with respect to $\mathcal{D}_2$?". We then try to answer that query by searching for a strategy with respect to $\mathcal{D}_2$, that is, by employing definitional reasoning (in addition to purely logical reasoning). Finding a strategy means that the query has a positive answer. In addition, one can in general gain further information from the variable bindings which have been computed in the construction of a strategy.

2.7 Definitional dialogues and contraction

In the following, we consider the paradoxical definitional clause

$$a \Leftarrow \neg a$$

(using $\neg A := A\to\bot$ for all formulas A, this is just an abbreviation for $a\Leftarrow a\to\bot$ here). It is related to Curry's Paradox—respectively to one of its special cases, namely Russell's Paradox—where for $t \in \{x \mid A\} \Leftarrow A[t/x]$ and $t = \{x \mid \neg(x \in x)\}$ with $A = \neg(x \in x)$ we have $t \in t \Leftarrow \neg(t \in t)$. The latter clause is of the form $a \Leftarrow \neg a$.

Example 2.18 For the given definitional clause $a \Leftarrow \neg a$ there is a strategy for a as well as for $\neg a$:

0.	$P\,a$	
1.	$O\,\mathscr{D}$	$[0, A]$
2.	$P\,\neg a$	$[1, D]$
3.	$O\,a$	$[2, A]$
4.	$P\,\mathscr{D}$	$[3, A]$
5.	$O\,\neg a$	$[4, D]$
6.	$P\,a$	$[5, A]$

0.	$P\,\neg a$	
1.	$O\,a$	$[0, A]$
2.	$P\,\mathscr{D}$	$[1, A]$
3.	$O\,\neg a$	$[2, D]$
4.	$P\,a$	$[3, A]$

Remark 2.19 These two strategies correspond to the following two natural deduction derivations for the given definitional clause $a \Leftarrow \neg a$, respectively:

$$\dfrac{\dfrac{\dfrac{[a]^2 \qquad \dfrac{[a]^2 \quad [\neg a]^1}{\bot}\,(\to \mathrm{E})}{\bot}\,(\text{def. reflection})^1}{\neg a}\,(\to \mathrm{I})^2}{a}\,(\text{def. closure}) \qquad\qquad \dfrac{\dfrac{[a]^2 \qquad \dfrac{[a]^2 \quad [\neg a]^1}{\bot}\,(\to \mathrm{E})}{\bot}\,(\text{def. reflection})^1}{\neg a}\,(\to \mathrm{I})^2$$

(Where again $\neg a := a \to \bot$.)

Remark 2.20 The existence of a strategy for a as well as for $\neg a$ in Example 2.18 depends on the fact that in the last move the proponent P can state the formula a (in the moves $\langle\delta(6) = P\,a, \eta(6) = [5, A]\rangle$ and $\langle\delta(4) = P\,a, \eta(4) = [3, A]\rangle$, respectively), which has been attacked by P with definitional reasoning before (in the moves $\langle\delta(4) = P\,\mathscr{D}, \eta(4) = [3, A]\rangle$ and $\langle\delta(2) = P\,\mathscr{D}, \eta(2) = [1, A]\rangle$, respectively).

That a is stated in the last move of a dialogue in a strategy means that a is used without reference to its definition, like the assumption a used as minor premiss in the inference $(\to \mathrm{E})$ of the corresponding natural deduction derivations.

However, here this move is possible only after having reflected on the definition of a by definitional reasoning; this corresponds to the use of the assumption a as the major premiss (i.e. the left premiss) in the inference of definitional reflection in the natural deduction derivations. Hence, the formula a has been used both with and without referring to its definition. This means that the differently used occurrences of the formula a have been *contracted* implicitly.

In other words, the proponent P has not only made twofold use of the formula a (asserted by O at position 3) in the moves at positions 4 and 6 of the left dialogue, respectively in the moves at positions 2 and 4 of the right dialogue (i.e., contractions of the form (3) as given in Remark 1.22), but the formula a has also been used in two different senses: once as an arbitrary assumption and once according to its given definition.

Remark 2.21 One way to avoid paradoxes of the above kind lies thus in restricting the structural operation of contraction in a suitable way. Disallowing contraction altogether would be too strong, since there would then no longer be strategies for formulas like $\neg(a \wedge \neg a)$ and $\neg\neg(a \vee \neg a)$ (cf. Example 1.23). What is needed is a restriction of contraction to only such occurrences of formulas which are not used in different senses.

Third Lecture: Dialogues for Implications as Rules

3.1 Introduction

In the last lecture we have considered extensions of logic by a certain kind of definitions. These definitions $\mathcal{D}$ are finite sets of definitional clauses for atomic formulas whose defining conditions can be any (atomic or complex) formula. They have the general form

$$\mathcal{D}\begin{cases} a_1 \Leftarrow \Gamma_1^1 \\ \quad\vdots \\ a_1 \Leftarrow \Gamma_{k_1}^1 \\ \quad\vdots \\ a_n \Leftarrow \Gamma_1^n \\ \quad\vdots \\ a_n \Leftarrow \Gamma_{k_n}^n \end{cases}$$

where the a_i are atomic formulas and the $\Gamma_{l_i}^i$ are (atomic or complex) formulas. The definitional clauses

$$a_i \Leftarrow \Gamma_{l_i}^i$$

in such definitions can also be read as rules

$$\frac{\Gamma_{l_i}^i}{a_i}$$

Indeed, with the principle of definitional closure, definitional clauses were used as rules. In addition, we have used the principle of definitional reflection. Both principles were incorporated in the argumentation form of definitional reasoning, and with definitional dialogues a dialogical framework was formulated for definitional reasoning about clausal definitions. It can be observed that definitional clauses are very similar to implicative statements. The theoretical basis for definitional dialogues were EI_c^p-dialogues, which—when won by the proponent P—need not end with the assertion of an atomic formula, but can also end with the assertion of a complex formula. This was effected by the following dialogue condition:

($D14$) O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

In this third lecture we want to reconsider the meaning of the logical constant of implication '$\to$' by interpreting implications $A \to B$ as rules. For sequent calculus, Schroeder-Heister [2011] has introduced an alternative left introduction rule for implication, which is motivated by the interpretation of implications as rules. Here we will look at its dialogical counterpart by giving a dialogical framework for implications as rules (see also Piecha and Schroeder-Heister [2012]).

3.2 Implications as rules

Usually, constructive interpretations of implication are more or less directly given by the Brouwer–Heyting–Kolmogorov (BHK) interpretation, according to which a proof of an implication $A \to B$ consists of a construction transforming any given proof of A into a proof of B; in the formulation of Heyting [1971, p. 102f.]:

> The *implication* $\mathfrak{p} \to \mathfrak{q}$ can be asserted, if and only if we possess a construction $\mathfrak{r}$, which, joined to any construction proving $\mathfrak{p}$ (supposing that the latter be effected), would automatically effect a construction proving $\mathfrak{q}$. In other words, a proof of $\mathfrak{p}$, together with $\mathfrak{r}$, would form a proof of $\mathfrak{q}$.

The standard dialogical interpretation of implication is based on the same idea: An implication $A \to B$ is attacked by claiming A and defended by claiming B. In order to have a strategy for $A \to B$, the proponent must be able to produce a substrategy (cf. Definition 3.8 below) for B from what the opponent uses in defending A. A difference to standard constructive interpretations is that the opponent need not necessarily give a full proof of A which is then transformed into a proof of B. Instead, the proponent may force the opponent to produce certain fragments of a proof of A that are sufficient to produce a substrategy for B.

A more elementary view of implication is based on the conception that an implication $A \to B$ is a rule which allows one to pass over from A to B. This view is particularly supported by the treatment of implication in natural deduction. There *modus ponens* (i.e., implication elimination ($\to$E))

$$\frac{A \quad A \to B}{B}$$

can be read as the application of $A \to B$ as a rule, which is used to infer B from A, that is, *modus ponens* can be read as a schema of rule application:

$$\frac{A}{B}\,(A \to B)$$

The introduction of an implication $A \to B$ by

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}$$

(where assumptions A can be discharged) can be read as establishing a rule, namely by deriving its conclusion B from its premiss A. Applications of logic such as logic programming or definitional reasoning support this approach.

When implications are read as rules, an elementary meaning is given to implication which is conceptually prior to the meaning of the other logical constants (see Schroeder-Heister [2011]).

In the following, we explain how the implications-as-rules approach can be carried over to dialogues. This is done in two steps: We first introduce preliminary EI°-dialogues, which implement the implications-as-rules approach. These preliminary dialogues will be found lacking, since they are not sufficient for intuitionistic logic. In the second step, we correct this by making an addition to the preliminary EI°-dialogues, yielding EI°-dialogues for intuitionistic logic. We will only treat the propositional case; the results can be generalized to the first-order case. Our approach is again constructive, respectively intuitionistic.

3.3 Preliminary EI°-dialogues

The guiding idea for implications-as-rules dialogues is the following: Once an implication $C \to A$ has been claimed by the opponent, it is considered to be a rule in a kind of 'database', which later can be used by the proponent to reduce the justification of its conclusion A to the justification of its premiss C. This is achieved by allowing the proponent to defend an attack on A by asserting C whenever $C \to A$ has been claimed by the opponent before. In case no such claim has been made before (i.e., if no applicable rule is available in the database), the argument for A continues as usual with an opponent attack on A (which must eventually be defended by the proponent), depending on the respective form of A. When making an assertion A, the proponent P must be prepared to either defend A in the 'standard' way against an attack of the opponent O, or else make the assertion C for some C, for which O has already claimed $C \to A$, that is, for which the implication-as-rule $C \to A$ is sufficient to generate A. This is implemented by saying that every assertion made by P is symbolically questioned by O, following which P chooses which of the two ways described P is prepared to take. Contrary to the proponent P, the opponent O is not given a choice. The opponent's non-implicational assertions are attacked and defended as usual, whereas the opponent's implicational assertions are considered as providing rules which the proponent can use, but not question; so there are no attacks and defenses defined for them.

Definition 3.1 For each logical constant we first define *argumentation forms* which determine how a complex formula (having the respective constant in outermost position) that has been asserted by the opponent O can be attacked (if possible) and how this attack can be defended (if possible): *argumentation forms*

AF($\wedge\vdash$):	assertion:	$O\, A_1 \wedge A_2$	
	attack:	$P \wedge_i$	(P chooses $i = 1$ or $i = 2$)
	defense:	$O\, A_i$	
AF($\vee\vdash$):	assertion:	$O\, A_1 \vee A_2$	
	attack:	$P \vee$	
	defense:	$O\, A_i$	(O chooses $i = 1$ or $i = 2$)

$\mathrm{AF}(\to\,\vdash)^\circ$: assertion: $O\,A \to B$
attack: *no attack*
defense: *no defense*

$\mathrm{AF}(\neg\,\vdash)$: assertion: $O\,\neg A$
attack: $P\,A$
defense: *no defense*

Except for $\mathrm{AF}(\to\,\vdash)^\circ$, these argumentation forms coincide with the standard ones (cf. Definition 1.2) in case of assertions made by the opponent O. (The argumentation form $\mathrm{AF}(\to\,\vdash)^\circ$ could also be omitted, to the same effect. However, we prefer to give the argumentation form $\mathrm{AF}(\to\,\vdash)^\circ$ in order to make it explicit that implications $A \to B$ asserted by O cannot be attacked.)

We now extend our language by the two special symbols ? and $|\cdot|$. For assertions made by the proponent P there is a pair of argumentation forms for each logical constant (depicted below as trees having two branches which are separated by $|$). An assertion A made by the proponent P can be questioned by the opponent with the move $O\,?$ (such a move is only possible if the expression stated in the P-move is an assertion, that is, a formula; if it is not an assertion but a symbolic attack, then it cannot be questioned with the move $O\,?$).

The proponent P can then answer this question either by allowing an attack on the assertion (this is indicated by the special symbol $|\cdot|$; see the argumentation forms on the left side of $|$ below), or by asserting any formula C for which O has asserted the implication $C \to A$ at an earlier position. We call this the *rule condition* (R): *rule condition*

(R) P may answer a question $O\,?$ on a formula A by choosing C provided O has asserted the formula $C \to A$ before.

The argumentation forms for assertions made by the proponent P are then defined as follows:

$\mathrm{AF}(\vdash\,\wedge)$:

assertion:	$P\,A_1 \wedge A_2$	
question:	$O\,?$	
choice:	$P\,\lvert A_1 \wedge A_2\rvert$	$P\,C$ (R)
attack:	$O\,\wedge_i$ ($i = 1$ or 2)	
defense:	$P\,A_i$	

$\mathrm{AF}(\vdash\,\vee)$:

assertion:	$P\,A_1 \vee A_2$	
question:	$O\,?$	
choice:	$P\,\lvert A_1 \vee A_2\rvert$	$P\,C$ (R)
attack:	$O\,\vee$	
defense:	$P\,A_i$ ($i = 1$ or 2)	

$\mathrm{AF}(\vdash\,\to)$:

assertion:	$P\,A \to B$	
question:	$O\,?$	
choice:	$P\,\lvert A \to B\rvert$	$P\,C$ (R)
attack:	$O\,A$	
defense:	$P\,B$	

AF($\vdash \neg$):	assertion:		$P\,\neg A$	
	question:		$O\,?$	
	choice:	$P\,\lvert\neg A\rvert$	$P\,C$	(R)
	attack:	$O\,A$		
	defense:	*no defense*		

In the case of an attack $O \wedge_i$ according to the argumentation form AF($\vdash \wedge$) for conjunctive formulas asserted by P, the opponent O chooses $i = 1$ or $i = 2$, and in the case of a defense $P\,A_i$ to an attack $O \vee$ according to the argumentation form AF($\vdash \vee$) for disjunctive formulas asserted by P, the proponent P chooses $i = 1$ or $i = 2$. The argumentation forms on the left (i.e., the respective left branches) correspond to the argumentation forms given in Definition 1.2 for 'standard' dialogues (where the device of question and choice moves is not needed). The argumentation forms on the right (i.e., the respective right branches) reflect the implications-as-rules view.

For assertions of atomic formulas a made by the proponent P an argumentation form is given by the rule condition (R) itself:

AF(R):	assertion:	$P\,a$	
	question:	$O\,?$	
	choice:	$P\,C$	only if O has asserted $C \to a$ before

Remark 3.1 In Definition 1.2 argumentation forms were defined independently of whether the assertion is made by the proponent P or by the opponent O. This symmetry is not preserved here.

Definition 3.2 We extend the definition of *moves* (see Definition 1.3) as follows: *moves*

As before, pairs $\langle \delta(n), \eta(n) \rangle$ are called *moves*, where $\delta(n)$, for $n \geq 0$, is again a signed expression, and $\eta(n)$ is again a pair $[m, Z]$, for $0 \leq m < n$, where Z is now either A (for 'attack'), D (for 'defense'), Q (for 'question') or C (for 'choice'). As before, $\eta(n) = [m, Z]$ is empty for $n = 0$, that is, $\eta(0) = \emptyset$.

We have thus the following types of moves:

$$\begin{array}{rl} \textit{attack move} & \langle \delta(n) = X\,e, \eta(n) = [m, A] \rangle, \\ \textit{defense move} & \langle \delta(n) = X\,A, \eta(n) = [m, D] \rangle, \\ \textit{question move} & \langle \delta(n) = O\,?, \eta(n) = [m, Q] \rangle, \\ \textit{choice move} & \langle \delta(n) = P\,|A|, \eta(n) = [m, C] \rangle, \\ & \langle \delta(n) = P\,A, \eta(n) = [m, C] \rangle. \end{array}$$

Remark 3.2 A question move can only be made by O, and a choice move can only be made by P. The other types of moves are available for both the proponent P and the opponent O.

In a choice move, $\delta(n)$ can have the form $P\,|A|$ or $P\,A$. In the first case, P allows an attack on the formula A. In the second case, P asserts the formula A in accordance with the rule condition (R), that is, A is the antecedent of an implication asserted by O before.

Dialogues for the implications-as-rules approach can now be defined as follows.

Definition 3.3 A *preliminary EI°-dialogue* is a sequence of moves $\langle\delta(n), \eta(n)\rangle$ $(n = 0, 1, 2, \ldots)$ satisfying the following conditions:

preliminary EI°-dialogue

($D00'$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01^\circ$) If $\eta(n) = [m, A]$ for even n, then the expression in $\delta(m)$ is a complex formula. If $\eta(n) = [n-1, A]$ for odd n, then the expression in $\delta(n-1)$ is of the form $|B|$ for a complex formula B. In both cases $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D03^\circ$) If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D or C, and the expression in $\delta(n)$ is the question mark '?'.

($D04^\circ$) If $\eta(n) = [m, C]$ (for even n), then $\eta(m) = [l, Q]$ for $l < m < n$ and $\delta(n)$ is the choice answering the question $\delta(m)$ as determined by the relevant argumentation form.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n'-n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.

That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd n there is at most one m such that $\eta(m) = [n, D]$.

That is, an attack by O may be defended by P at most once.

($D14'$) O can question a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n-1, Z]\rangle$, for $Z = Q$, A or D.

That is, an opponent move made at position n is either a question, an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for DI^p-dialogues are directly carried over to the corresponding notions for (preliminary) EI°-dialogues.

Remark 3.3 Preliminary EI°-dialogues are similar to EI^p_c-dialogues without condition ($D10$) for the argumentation forms given in Definition 3.1 and satisfying the condition ($D14'$) instead of ($D14$), where ($D14'$) differs from ($D14$) only in that the latter is a condition for O attacking a formula C, whereas the former is a condition for O questioning a formula C.

Condition ($D00'$) is the same as for definitional dialogues (cf. Definition 2.10). It allows (preliminary) EI°-dialogues to start with the assertion of an atomic formula, contrary to the restriction to complex formulas as, for example, in EI_c^p-dialogues (cf. Definition 2.3). Condition ($D01°$) differs from condition ($D01$) in EI_c^p-dialogues in that it allows for attacks by O on expressions of the form $|A|$ for complex formulas A. Condition ($D02$) is as given in Definition 1.4 for dialogues. Conditions ($D03°$) and ($D04°$) have been added for the question and choice moves, respectively.

We recall (cf. Remark 2.3) that condition (E) implies condition ($D13$), and that (E) also implies condition ($D11$) for odd p and condition ($D12$) for odd n. The conditions ($D11$) and ($D12$) have thus been weakened here to the conditions ($D11'$) and ($D12'$), respectively.

Remark 3.4 The absence of condition ($D10$) in the definition of preliminary $EI°$-dialogues is compensated for by allowing the opponent O to question assertions of atomic formulas made by the proponent P. In dialogues with ($D10$) there is, for example, no strategy for the formula $a \to b$, since the dialogue

0.	$P\,a \to b$	
1.	$O\,a$	$[0, A]$

cannot be continued with the move $\langle \delta(2) = P\,b, \eta(2) = [1, D] \rangle$; this would only be possible if b were asserted by O before.

In (preliminary) $EI°$-dialogues (where ($D10$) is absent) there is no strategy for $a \to b$ either. The (preliminary) $EI°$-dialogue begins with the moves

0.	$P\,a \to b$	
1.	$O\,?$	$[0, Q]$
2.	$P\,\lvert a \to b \rvert$	$[1, C]$
3.	$O\,a$	$[2, A]$
4.	$P\,b$	$[3, D]$
5.	$O\,?$	$[4, Q]$

where P can now assert b at position 4 without O having asserted it before. However, the opponent O can make a question move at position 5, in accordance with the argumentation form AF(R). The proponent P cannot make the choice move $\langle \delta(6) = P\,|b|, \eta(6) = [5, C] \rangle$ here, since there is no such argumentation form for atomic formulas. The only possible choice move would be one according to the argumentation form AF(R), that is, a move of the form $\langle \delta(6) = P\,C, \eta(6) = [5, C] \rangle$ for a formula $C \to b$ asserted by the opponent O before. But such a formula has not been asserted by O in this dialogue.

Remark 3.5 Due to condition ($D14'$), (preliminary) $EI°$-dialogues won by P need not end with the assertion of an atomic formula but can end with the assertion of a complex formula.

For example, the following dialogue is a (preliminary) $EI°$-strategy for the formula $(a \vee b) \to \neg\neg(a \vee b)$:

0.	$P\,(a \vee b) \to \neg\neg(a \vee b)$	
1.	$O\,?$	$[0, Q]$
2.	$P\,\lvert (a \vee b) \to \neg\neg(a \vee b) \rvert$	$[1, C]$

(continued on next page)

3.	$O\, a \vee b$	$[2, A]$
4.	$P\, \neg\neg(a \vee b)$	$[3, D]$
5.	$O\, ?$	$[4, Q]$
6.	$P\, \|\neg\neg(a \vee b)\|$	$[5, C]$
7.	$O\, \neg(a \vee b)$	$[6, A]$
8.	$P\, a \vee b$	$[7, A]$

The opponent O cannot question $a \vee b$, since neither of the two conditions (i) and (ii) of $(D14')$ is satisfied: $a \vee b$ has already been asserted by O at position 3, and $a \vee b$ has not been attacked by P.

Example 3.6 The following dialogue is a (preliminary) *EI*°-strategy for the formula $(a \to b) \to ((b \to c) \to (a \to c))$:

0.	$P\, (a \to b) \to ((b \to c) \to (a \to c))$		
1.	$O\, ?$	$[0, Q]$	
2.	$P\, \|(a \to b) \to ((b \to c) \to (a \to c))\|$	$[1, C]$	
3.	$O\, a \to b$	$[2, A]$	"assuming $a \to b$ as a rule"
4.	$P\, (b \to c) \to (a \to c)$	$[3, D]$	
5.	$O\, ?$	$[4, Q]$	
6.	$P\, \|(b \to c) \to (a \to c)\|$	$[5, C]$	
7.	$O\, b \to c$	$[6, A]$	"assuming $b \to c$ as a rule"
8.	$P\, a \to c$	$[7, D]$	
9.	$O\, ?$	$[8, Q]$	
10.	$P\, \|a \to c\|$	$[9, C]$	
11.	$O\, a$	$[10, A]$	
12.	$P\, c$	$[11, D]$	
13.	$O\, ?$	$[12, Q]$	
14.	$P\, b$	$[13, C]$	"using $b \to c$ as a rule"
15.	$O\, ?$	$[14, Q]$	
16.	$P\, a$	$[15, C]$	"using $a \to b$ as a rule"

At position 3, the opponent asserts the implication $a \to b$. The formula b, which occurs also as the succedent of this implication, is questioned at position 15. In accordance with the rule condition (R), the proponent asserts a—the antecedent of the implication—in the last move; the opponent cannot question this move due to condition $(D14')$.

The implication $b \to c$ is asserted by O in the move at position 7. The opponent questions c at position 13, which enables P to answer according to the rule condition (R) with the choice move $P\, b$ at position 14.

The implications $a \to b$ and $b \to c$ have thus been used as rules: the latter implication-as-rule allowed P to answer the question on c with b, and the former allowed P to answer the question on b with a.

3.4 *EI*°-dialogues with cut

For the preliminary *EI*°-dialogues considered so far, there is no strategy for the formula $a \to ((a \to (b \wedge c)) \to b)$. Consider the following dialogue:

0. $P\, a \to ((a \to (b \wedge c)) \to b)$

(*continued on next page*)

1.	$O\,?$	$[0, Q]$
2.	$P\,\lvert a \to ((a \to (b \wedge c)) \to b)\rvert$	$[1, C]$
3.	$O\,a$	$[2, A]$
4.	$P\,(a \to (b \wedge c)) \to b$	$[3, D]$
5.	$O\,?$	$[4, Q]$
6.	$P\,\lvert(a \to (b \wedge c)) \to b\rvert$	$[5, C]$
7.	$O\,a \to (b \wedge c)$	$[6, A]$
8.	$P\,b$	$[7, D]$
9.	$O\,?$	$[8, Q]$

The moves at positions 0–4 and at positions 4–7 + 12 are made according to the argumentation form AF($\vdash \to$). In the choice moves at positions 2 and 6 the proponent P can only choose $\lvert a \to ((a \to (b \wedge c)) \to b)\rvert$ and $\lvert(a \to (b \wedge c)) \to b\rvert$, respectively, since O has not asserted any implications before which could be used as rules by choosing their antecedents. At position 7 the opponent asserts the implication $a \to (b \wedge c)$. At position 8 the proponent P defends the attack $O\,a \to (b \wedge c)$ by asserting b; assertions by P of atomic formulas not asserted by O before are not prohibited in (preliminary) EI°-dialogues (they would be prohibited by condition ($D10$), for example in EI^{p}_{c}-dialogues). This move can be questioned by O at position 9, and P loses this dialogue, since P cannot make another move at position 10:

(i) P can neither choose $\lvert b\rvert$ nor C, since no move $O\,C \to b$ has been made for such a formula C,

(ii) there is no attack for $O\,a \to (b \wedge c)$ (by definition of AF($\to\vdash$)$^{\circ}$),

(iii) and for a being atomic there is no attack for the move $O\,a$ made at position 3.

Although there is no preliminary EI°-strategy, there is an EI^{p}_{c}-strategy for the formula $a \to ((a \to (b \wedge c)) \to b)$:

0.	$P\,a \to ((a \to (b \wedge c)) \to b)$	
1.	$O\,a$	$[0, A]$
2.	$P\,(a \to (b \wedge c)) \to b$	$[1, D]$
3.	$O\,a \to (b \wedge c)$	$[2, A]$
4.	$P\,a$	$[3, A]$
5.	$O\,b \wedge c$	$[4, D]$
6.	$P\,\wedge_1$	$[5, A]$
7.	$O\,b$	$[6, D]$
8.	$P\,b$	$[3, D]$

Thus preliminary EI°-dialogues cannot be complete for intuitionistic logic (contrary to EI^{p}_{c}-dialogues, for which we have Theorem 2.5).

In order to achieve completeness of the dialogical implications-as-rules framework for intuitionistic logic we have to add a form of cut to our preliminary EI°-dialogues. We first define an argumentation form for cut, extend our definition of moves for cut moves, and adjust our definition of preliminary EI°-dialogues accordingly, yielding the final definition of EI°-dialogues.

The implications-as-rules approach as such is independent of the presence of cut. However, cut moves have to be allowed if not only a fragment of intuitionistic (propositional) logic is to be captured.

Definition 3.4 We define an *argumentation form* AF(Cut) such that any expression e (i.e., question, symbolic attack or formula) stated by O can be followed by a move $P\,A$, and this move can then be followed by the opponent move $O\,A$: *argumentation form for cut*

AF(Cut):	statement:	$O\,e$
	cut:	$P\,A$
	cut:	$O\,A$

The formula A is called *cut formula* in this argumentation form. *cut formula*

Remark 3.7 The argumentation form AF(Cut) differs from the other argumentation forms in that the move $O\,e$ need not be an assertion (i.e., the statement of a formula) but can be the statement of any expression e (i.e., question, symbolic attack or formula).

Another difference is that the cut formula is completely independent of the expression e. Calling the P-move an attack and the subsequent O-move a defense as in the other argumentation forms would thus be inadequate. We therefore simply speak of *cut moves* in both cases.

The idea behind AF(Cut) is that at any (even) position the proponent P can introduce an arbitrary formula A as a lemma. The proponent P must then later be prepared both to defend this lemma A as an assertion and to defend the original claim (i.e., the assertion made in the initial move at position 0) given this lemma, that is, given the opponent's claim of A.

Definition 3.5 We extend the definition of *moves* (see Definition 3.2) further by adding the following type of move: *cut move*

$$\textit{cut move} \quad \langle \delta(n) = X\,A, \eta(n) = [\mathit{Cut}] \rangle.$$

(Note that here in the pair $\eta(n) = [m, Z]$, $Z = \mathit{Cut}$ and m is empty.)

Definition 3.6 *EI°-dialogues* are preliminary EI°-dialogues with the following additional dialogue condition ($D05^{\circ}$) and two small adjustments in conditions ($D03^{\circ}$) and (E) for cut moves: *EI°-dialogues*

($D03^{\circ}$) If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D, C or Cut (where l is empty if $Z = \mathit{Cut}$), and the expression in $\delta(n)$ is the question mark '?'.

($D05^{\circ}$) If $\eta(n) = [\mathit{Cut}]$ for even n, then $\eta(m) = [l, Z]$ (where l is empty if $Z = \mathit{Cut}$) for $l < m < n$ and $\delta(n)$ is a formula (i.e., the cut formula). If $\eta(n) = [\mathit{Cut}]$ for odd n, then $\eta(m) = [\mathit{Cut}]$ and $\delta(n) = O\,A$ for $\delta(m) = P\,A$ (where $m < n$).

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n - 1, Z] \rangle$, for $Z = Q$, A, D or Cut (where $n - 1$ is empty if $Z = \mathit{Cut}$).

That is, an opponent move made at position n is either a question, an attack or a defense of the immediately preceding move made by the proponent at position $n-1$, or it is a cut move with $\delta(n) = O\,A$ for $\delta(n-1) = P\,A$.

Definition 3.7 A formula A is called *EI°-dialogue-provable* (short: *EI°-provable*) if there is an *EI°*-strategy for A. Notation: $\vdash_{EI^\circ} A$.

EI°-dialogue-provable

Example 3.8 In the presence of cut, there is an *EI°*-strategy for the formula $a \to ((a \to (b \wedge c)) \to b)$:

0.	$P\, a \to ((a \to (b \wedge c)) \to b)$			
1.	$O\, ?$	$[0, Q]$		
2.	$P\, \lvert a \to ((a \to (b \wedge c)) \to b)\rvert$	$[1, C]$		
3.	$O\, a$	$[2, A]$		
4.	$P\, (a \to (b \wedge c)) \to b$	$[3, D]$		
5.	$O\, ?$	$[4, Q]$		
6.	$P\, \lvert(a \to (b \wedge c)) \to b\rvert$	$[5, C]$		
7.	$O\, a \to (b \wedge c)$	$[6, A]$		
8.	$P\, b \wedge c$	$[Cut]$		
9.	$O\, ?$	$[8, Q]$	$O\, b \wedge c$	$[Cut]$
10.	$P\, a$	$[9, C]$	$P \wedge_1$	$[9, A]$
11.			$O\, b$	$[10, D]$
12.			$P\, b$	$[7, D]$

Instead of defending the opponent's attack $a \to (b \wedge c)$ made at position 7, the proponent continues by asserting the succedent $b \wedge c$ of that implication in the cut move at position 8. It is questioned at position 9 (in the left dialogue). In accordance with the rule condition (R), the proponent can now answer this question move by asserting in the choice move at position 10 (in the left dialogue) the antecedent a of the implication whose succedent has been questioned. The implication $a \to (b \wedge c)$ asserted by O at position 7 was thus used as a rule. The opponent cannot question the formula a due to condition $(D14')$: O has already asserted a (in the attack move at position 3), and P has not attacked a (such an attack is not even possible, since a is atomic).

In the right dialogue, the opponent makes the corresponding cut move at position 9, which is attacked by P and defended by O with the assertion of b. Now P can defend the opponent's attack from position 7 by asserting b; as O has already asserted b without b having been attacked by P, the opponent O cannot question b due to condition $(D14')$, and the proponent P also wins the right dialogue.

3.5 Completeness

Completeness for *EI°*-dialogues and intuitionistic logic can be proved by showing that there is an *EI°*-strategy for a formula A if and only if there is an EI^p_c-strategy for A (see Theorem 3.12 below). Completeness (see Corollary 3.13 below) then follows from our completeness result for EI^p_c-dialogues (see Theorem 2.5).

Definition 3.8 A *substrategy* is a subtree s of a dialogue tree t comprising as root node a node at an even position in t and all descendents in t such that *substrategy*

(i) s does not branch at even positions,

(ii) s has as many nodes at odd positions as there are possible moves for O,

(iii) and all leaves are proponent moves such that O cannot make another move.

Lemma 3.9 *(i) The weak cut elimination property holds for EI°-strategies. That is, every EI°-strategy containing cut moves made according to the argumentation form* AF(Cut) *can be transformed into an EI°-strategy of the form* *weak cut elimination property*

$$
\begin{array}{rlll}
 & & \vdots & \\
m. & & O\,A \to B & [m-1, Z] \\
 & & \vdots & \\
n. & & P\,B & [Cut] \\
n+1. & O\,? \ [n, Q] & \multicolumn{2}{|l}{O\,B \ [Cut]} \\
n+2. & P\,A \ [n+1, C] & \multicolumn{2}{|l}{\quad s_2} \\
n+3. & O\,? \ [n+2, Q] & \multicolumn{2}{|l}{} \\
 & \quad s_1 & \multicolumn{2}{|l}{}
\end{array}
$$

where the O-move at position m is either an attack or a defense (i.e., either $Z = A$ or $Z = D$), and the move $\langle \delta(n+1) = O\,B, \eta(n+1) = [Cut] \rangle$ is the uppermost cut move made by O (i.e., there is no cut move at positions $k < n-1$). The O-move at position $n+3$ might not be possible due to $(D14')$. In this case the left dialogue ends with the P-move at position $n+2$.

(Note that the cut formula B is a subformula of $A \to B$, asserted by O at position m.)

(ii) Furthermore, the substrategy s_2 is either of the same form as the above EI°-strategy, or it depends on a sequence of moves made according to AF($\wedge\vdash$), AF($\vee\vdash$), AF($\to\vdash$)° *or* AF($\neg\vdash$).

Corollary 3.10 *As a consequence of the weak cut elimination property, EI°-strategies have the subformula property. (This is in full analogy to the results on the weak cut elimination property and the subformula property for sequent calculus derivations with the alternative left implication introduction rule $(\to\vdash)^\circ$; cf. Schroeder-Heister [2011].)* *subformula property*

Lemma 3.11 *(i) EI°-strategies for formulas of the form*

$$A \to ((A \to (B \wedge C)) \to B)$$

containing a cut move where the cut formula is of the form $B \wedge C$ cannot be transformed into EI°-strategies (for the respective formula) containing no cut move. However, they can be transformed into EI^p_c-strategies (for the respective formula).

(ii) Every other EI°-strategy (for a given formula) containing a cut move can be transformed into an EI^p_c-strategy (for the given formula) as well.

Theorem 3.12 *There is an EI°-strategy for a formula A if and only if there is an EI^p_c-strategy for A, that is, $\vdash_{EI^\circ} A$ if and only if $\vdash_{EI^p_c} A$.*

Corollary 3.13 (Completeness) *With Theorem 2.5 we have that the EI°-provable formulas are exactly the formulas provable in intuitionistic logic.*

3.6 Discussion

One of the main differences between standard dialogues (like DI^p- or EI^p-dialogues) and EI°-dialogues is that the argumentation forms in the latter are no longer symmetric with respect to proponent and opponent; that is, the player independence of the argumentation forms that obtains in the standard dialogues is given up in EI°-dialogues: different argumentation forms have to be given for proponent and opponent. Although in standard dialogues proponent and opponent are also not interchangeable due to the dialogue conditions (cf. Remark 1.7), there is a perfect symmetry with respect to the argumentation forms. If the idea of having player independent argumentation forms is considered to be essential in the dialogical paradigm, then giving it up may seem to amount to giving up the dialogical setting itself as a foundational approach. However, from the implications-as-rules point of view it could be argued that implication is different from the other logical constants, and that this difference requires an asymmetric treatment with respect to the argumentation forms.

As a consequence of this asymmetry in the treatment of implication there is another asymmetry: In EI°-dialogues the proponent can defend an assertion by means of the rule condition (R) independently of its logical form. This is not possible in standard dialogues where a defense of an assertion always depends on its logical form, and where formulas are always decomposed into subformulas according to their logical form. Nonetheless, we have shown that the subformula property holds at least for EI°-strategies.

But certain tenets within the dialogical tradition—such as the player independence of argumentation forms or the decomposition of formulas according to their logical form—do not have to be taken as being essential in dialogical approaches. Particularly not for implications as rules: Rules are *not* logical constants but belong to the general structural framework that underlies definitions or meaning explanations of logical constants. Given that the proponent has the dialogical role of claiming something to hold, and the opponent the role of providing the assumptions under which something is supposed to hold, the implication-as-rule $A \to B$ means for the proponent that B must be defended on the background A, whereas the opponent only grants with $A \to B$ the right to *use* this implication as a rule, without any propositional claim. This is exactly what is captured in the EI°-dialogues for implications-as-rules.

An important aspect here is the significance which is given to *modus ponens*. For the implications-as-rules view, *modus ponens* is essential for the meaning of implication as it expresses the idea of *application*, which is the characteristic feature of a rule. In natural deduction, *modus ponens* can be understood as the application of the implication $A \to B$ as a rule which allows us to infer B from A. In EI°-dialogues this means that a dialogue on B can be reduced to a dialogue on A, if an implication-as-rule $A \to B$ is given. We have thus an argumentative interpretation for implications as rules.

A further complication is introduced by the need of (a restricted form of) cut in order to achieve full intuitionistic logic. Although this need is present in both the proof-theoretic setting (e.g. using sequent calculus) and the dialogical setting for implications-as-rules, the addition of an argumentation form for cut might be conceived as being alien to the dialogical approach as such, as this approach has always been considered as being cut-free per se. But from the perspective of implications-as-rules such a view proves to be too narrow—at least if full intuitionistic logic is to be achieved.

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