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Proof-theoretic harmony and the levels of rules: Generalised non-flattening results

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Abstract

If we generate elimination from introduction rules, or, conversely, introduction rules from elimination rules according to a general pattern, we often observe a rise in level: To introduction rules that are just production rules, there correspond elimination rules that discharge assumptions, and vice versa. In a previous publication we showed that this situation cannot always be avoided, i.e., that elimination and introduction rules cannot always be 'flattened'. More precisely, we showed that there are connectives with given introduction rules which do not have corresponding elimination rules in standard natural deduction, and vice versa. In this paper we generalise this result: Even if we allow for rules of higher levels, i.e. rules that may discharge rules used as assumptions, the level rise is often necessary. For every level n we can find a connective with introduction rules of level n, whose corresponding elimination rules must at least have level $n+1$, and a connective with elimination rules of level n, whose corresponding introduction rules must at least have level $n+1$.

1 Introduction

Within proof-theoretic semantics (Schroeder-Heister, 2012; Wansing, 2000), various notions of harmony between introduction and elimination rules (in natural deduction), or between right-introduction and left-introduction rules (in the sequent calculus) have been proposed. The most common approaches in the natural-deduction framework

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proceed by presenting a general schema for elimination rules given introduction rules of a certain form (Prawitz, 1979; Schroeder-Heister, 1984; Read, 2010; Francez & Dyckhoff, 2012)[1]. These schemas have the characteristic feature that they model the rules for an arbitrary connective according to the rules for disjunction. Now the rules for disjunction

$$\frac{A_1}{A_1 \vee A_2} \quad \frac{A_2}{A_1 \vee A_2} \qquad \frac{A_1 \vee A_2 \quad \begin{array}{c}[A_1]\\ C\end{array} \quad \begin{array}{c}[A_2]\\ C\end{array}}{C}$$

are of unequal level in the sense that the introduction rules are just production rules, whereas the elimination rule discharges assumptions. Our question is, whether such unequal levels can be avoided, i.e., whether elimination rules can be 'flattened' in that they receive the level of introduction rules (the term 'flattening' has been proposed by Read 2014). That this is impossible in the case of disjunction (in intuitionistic logic!) is not a real problem, as both introduction and elimination rules for disjunction are perfectly sensible rules in natural deduction. However, when we consider connectives, whose introduction rules discharge assumptions, the flattening problem becomes significant, because it then turns into the problem of whether such connectives can be represented in standard natural deduction at all. In our first paper on flattening (Olkhovikov & Schroeder-Heister, 2014) we proved that the three-place connective $\star$ which has the introduction rules

$$(\star\ \mathrm{I}) \qquad \frac{\begin{array}{c}[A_1]\\ A_2\end{array}}{\star(A_1, A_2, A_3)} \quad \frac{A_3}{\star(A_1, A_2, A_3)} \tag{1}$$

cannot be given elimination rules in standard natural deduction, i.e. that the flattening problem for $\star$ has a negative solution[2]. This means that we cannot proof-theoretically characterise $\star$ by introduction and elimination inferences without presupposing any other connective. Of course, if implication and disjunction are already available, then $\star$ can be trivially characterised by explicitly defining it by $(A_1 \to A_2) \vee A_3$, or, equivalently, by giving it the introduction and elimination rules

$$\frac{(A_1 \to A_2) \vee A_3}{\star(A_1, A_2, A_3)} \qquad \frac{\star(A_1, A_2, A_3)}{(A_1 \to A_2) \vee A_3} \ .$$

Even though $\star$ cannot be proof-theoretically characterised in standard natural deduction, it can be characterised in an extension of natural deduction, in which not only

[1] Von Kutschera (1968) was the first to propose such a general schema, but in the framework of a sequent calculus. His schema can easily be carried over to natural deduction.

[2] A connective with the introduction rules of $\star$ is already mentioned in Zucker and Tragesser (1978). Related connectives have been discussed by von Kutschera (1968), Dyckhoff (2009, 2014), Schroeder-Heister (2014b) and Read (2014). However, none of these papers provided a formal proof that the flattening problem has a negative solution for the connective considered.

formulas, but also rules can figure as assumptions which can be discharged (Schroeder-Heister, 1984). In such a framework the elimination rule for $\star$ takes the form

$$(\star\,\mathrm{E})\qquad \frac{\star(A_1,A_2,A_3) \quad \begin{array}{c}[A_1 \Rightarrow A_2]\\ C\end{array} \quad \begin{array}{c}[A_3]\\ C\end{array}}{C}\ ,$$

where the bracketed $A_1 \Rightarrow A_2$ means that in the corresponding derivation of C the rule

$$\frac{A_1}{A_2}$$

may be used as an additional assumption which is discharged at the application of ($\star$ E).

A similar phenomenon occurs when we start from eliminations and try to formulate a general schema for introductions, even though this approach is not very common (see Prawitz, 1971, 2007; Dummett, 1991, Ch. 13; Schroeder-Heister, 2014a). If we start from modus ponens

$$\frac{A_1 \to A_2 \quad A_1}{A_2}$$

as the elimination rule for implication, we can generate implication introduction

$$\frac{\begin{array}{c}[A_1]\\ A_2\end{array}}{A_1 \to A_2}$$

by turning minor premiss and conclusion of the elimination into assumption and premiss, respectively, of the introduction. This follows a uniform procedure, by means of which we would, for example, for a four-place constant c_1 with the elimination rules

$$\frac{c_1(A_1,A_2,A_3,A_4) \quad A_1 \quad A_2}{A_3} \qquad \frac{c_1(A_1,A_2,A_3,A_4)}{A_4}$$

generate the following introduction rule

$$\frac{\begin{array}{c}[A_1,A_2]\\ A_3\end{array} \qquad A_4}{c_1(A_1,A_2,A_3,A_4)}\ .$$

However, whereas the rules for $\to$ and c_1 do not exceed the expressive power of standard natural deduction, the three-place connective $\circ$ with the single elimination rule

$$(\circ\,\mathrm{E})\qquad \frac{\circ(A_1,A_2,A_3) \quad \begin{array}{c}[A_1]\\ A_2\end{array}}{A_3} \tag{2}$$

would do so. As shown in Olkhovikov and Schroeder-Heister (2014, §4), it cannot be given an introduction rule in standard natural deduction, if we do not presuppose other connectives. (In terms of implication, $\circ$ would, of course, be definable by $(A_1 \to A_2) \to A_3$.) If flattening now means that the level of the introduction rules equals (or is below) the level of the eliminations rules, then $\circ$ with (2) as elimination rule is a counterexample to flattening.

Again, in an extension of natural deduction, in which rules can be assumptions, there is an appropriate introduction rule for $\circ$, namely the rule

$$(\circ\ \mathrm{I}) \qquad \frac{\begin{array}{c}[A_1 \Rightarrow A_2]\\ A_3\end{array}}{\circ(A_1, A_2, A_3)}\ . \tag{3}$$

In this paper we deal with this question from a more general point of view. As soon as we have introduced rules as assumptions, we can characterise further connectives by means of this general device. For example, we could give the four-place connective $\star_2$ the introduction rules

$$(\star_2\ \mathrm{I}) \qquad \frac{\begin{array}{c}[A_1 \Rightarrow A_2]\\ A_3\end{array}}{\star_2(A_1, A_2, A_3, A_4)} \qquad \frac{A_4}{\star_2(A_1, A_2, A_3, A_4)} \tag{4}$$

and ask for the means of expression needed for appropriate elimination rules for $\star_2$. Or we might consider the four-place connective $\circ_2$ with the elimination rule

$$(\circ_2\ \mathrm{E}) \qquad \frac{\circ_2(A_1, A_2, A_3, A_4) \qquad \begin{array}{c}[A_1 \Rightarrow A_2]\\ A_3\end{array}}{A_4}$$

and ask for the means of expression needed for appropriate introduction rules for $\circ_2$. Are there elimination rules for $\star_2$ or introduction rules for $\circ_2$ using rules of the form $B_1, \ldots, B_m \Rightarrow B$ as assumptions, or do we need even further means to formulate appropriate elimination rules for $\star_2$ and introduction rules for $\circ_2$? In other words, we are carrying over the flattening problem to a higher level. The purpose of this paper is to show that the flattening problem has always a negative solution. At any level n we can find a connective $\star_n$ with given introduction rules, for which there is demonstrably no appropriate set of elimination rules at the same (or lower) level, and a connective $\circ_n$ with given elimination rules, for which there is demonstrably no appropriate set of introduction rules at the same (or lower) level.

In Section 2 we define the extension of natural deduction with rules of higher levels and show that higher-level-rules correspond to conjunction-implication formulas of a certain form. This correspondence is essential to our technical work. Section 3 presents general schemas for introduction and elimination rules for an n-place (intuitionistic) connective, and defines what it means that introduction and elimination rules are in harmony with each other. This definition uses the framework of second-order intuitionistic propositional logic that we already used in our first paper (Olkhovikov & Schroeder-Heister, 2014), and which, as a 'reductive' approach to harmony, is discussed in Schroeder-Heister (2014c). Based on these definitions, we can formulate our main result, the generalised non-flattening claim for $\star_n$ and $\circ_n$. We reduce the proof that $\star_n$ and $\circ_n$ do not have harmonious elimination and introduction rules, respectively, which are of the same (or lower) level than their introduction and elimination rules to characteristic properties of certain formulas, called n-introduction and n-elimination formulas. In Section 4, which is the main technical part of this paper, we present the (somewhat intricate) formal proof that these formulas have the desired characteristic properties, which is an investigation in intuitionistic propositional logic. Section 5 is a concluding discussion of our results.

2 Higher-level rules

We work in the language L of an intuitionistic natural deduction calculus with propositional variables $p, q, r, \ldots$, with and without indices. The connectives available, from which formulas are composed, may be the standard connectives $\wedge, \vee, \rightarrow$ and $\bot$ or a subset thereof, but also n-place connectives yielding formulas of the form $c(A_1, \ldots, A_n)$. From the context it will always be clear, which connectives are considered. We use capital Latin letters $A, B, C, \ldots$, with and without indices, for formulas. Besides formulas, we have rules as separate entities. They are written linearly using the 'rule arrow' $\Rightarrow$. The rule which allows one to pass over from $A_1, \ldots, A_n$ to B is written as $A_1, \ldots, A_n \Rightarrow B$. In addition to rules as objects we shall define schemas for the application of rules. Such a schema tells what it means to apply a rule. For example, the schema

$$\frac{A_1 \ \ldots \ A_n}{B} \, A_1, \ldots, A_n \Rightarrow B$$

says that by applying the rule $A_1, \ldots, A_n \Rightarrow B$, we may pass from $A_1, \ldots, A_n$ to B. By means of such schemas we explain what a derivation looks like. However, we often identify a schema with the rule applied in it. For example, we speak of the $\vee$-introduction rule

$$\frac{A_1}{A_1 \vee A_2}$$

where we actually mean the rule $A_1 \Rightarrow A_1 \vee A_2$ which is applied according to the schema

$$\frac{A_1}{A_1 \vee A_2} \, A_1 \Rightarrow A_1 \vee A_2 \quad .$$

The duplicity of rules and schemas might be confusing. In standard natural deduction we can dispense with it by identifying rules with schemas throughout. But at soon as we want to use rules as expressions that we can assume and discharge, we need both rules as objects and schemas which tell one how rules are applied.[3]

We do not consider rules that allow one to infer rules, which means that rules can have only formulas as conclusions. For higher-level rules this has the effect that the rule arrow can only be iterated to the left, and that proper rules (i.e. rules which are not just formulas) only occur as assumptions and never as conclusions. This is essentially a matter of convenience, as rules as conclusions can easily be introduced by means of certain additional schemas. Philosophically, the idea that rules are always applied and never established, is nearer to the very idea of a rule and makes a rule distinct from an implication, which may have consequents which are themselves implications (see the discussion in Schroeder-Heister, 2014a).

In the following, when we say of formulas or rules that they are of *maximum* degree or *maximum* level n, we mean that the degree or level of these entities does not exceed n, but that n is reached by at least one of them. Rules of higher-levels are then defined as follows:

Definition 1.

- *Every formula A is a rule of level 0.*
- *If $R_1, \ldots, R_n$ are rules of maximum level ℓ, then $(R_1, \ldots, R_n \Rightarrow A)$ is a rule of level $\ell+1$. $R_1, \ldots, R_n$ are called the premisses, and the formula A the conclusion of the rule. Parentheses can be omitted, when no misreading is possible.*

A rule is assumed in a derivation by using it in the following way. This usage may itself discharge previous applications of other rules. If the rule is a formula A, then it is assumed by means of the schema

$$\frac{}{A}\,A \quad . \tag{5}$$

The subsequent derivation then *depends* on the rule A as an assumption. If it is of the form $A_1, \ldots, A_n \Rightarrow B$, then it is assumed by applying it according to the schema

$$\frac{A_1 \ \ldots \ A_n}{B}\,A_1, \ldots, A_n \Rightarrow B \quad . \tag{6}$$

The subsequent derivation then *depends* on the rule $A_1, \ldots, A_n \Rightarrow B$ as an assumption. If it is of the form $(\Delta_1 \Rightarrow A_1), \ldots, (\Delta_n \Rightarrow A_n) \Rightarrow B$, where each Δ_i stands for a list of rules (which may be empty, in which case $\Delta_i \Rightarrow A_i$ is identified with A_i), then it is applied according to the schema

$$\frac{\begin{matrix}[\Delta_1] \\ A_1\end{matrix} \qquad \ldots \qquad \begin{matrix}[\Delta_n] \\ A_n\end{matrix}}{B}\,(\Delta_1 \Rightarrow A_1), \ldots, (\Delta_n \Rightarrow A_n) \Rightarrow B \tag{7}$$

[3] In Schroeder-Heister (1984) a system with rules of higher levels was defined in which rules do not enter derivations as objects but can be reconstructed from proof trees. Here we are treating rules as formal objects which occur in proofs. The reader might consider the more detailed presentation in Schroeder-Heister (2014a), where in addition propositional quantification in rules is considered.

The subsequent derivation then *depends* on the rule $(\Delta_1 \Rightarrow A_1), \ldots, (\Delta_n \Rightarrow A_n) \Rightarrow B$ as an assumption, while the rules in the Δ_i above this rule application can be discharged (as indicated by the brackets), so that the subsequent derivation no longer depends on them. Schema (7) is understood as covering (5) and (6) as limiting cases.

A derivation is generated by applications of (7). If the undischarged assumptions on which a derivation of a formula A depends, are among $R_1, \ldots, R_n$, we call it a derivation of A from $R_1, \ldots, R_n$ and say that A is derivable from $R_1, \ldots, R_n$, symbolically $R_1, \ldots, R_n \vdash A$. For example, the following derivation demonstrates that $((A \Rightarrow B) \Rightarrow D), ((B, D) \Rightarrow C), (((A \Rightarrow B) \Rightarrow C) \Rightarrow E), ((B \Rightarrow E) \Rightarrow F) \vdash F$:

$$(3)\,\cfrac{(2)\,\cfrac{\cfrac{\cfrac{}{B}\,[B]^{(3)} \qquad (1)\,\cfrac{\cfrac{\cfrac{}{A}\,[A]^{(1)}}{B}\,[A \Rightarrow B]^{(2)}}{D}\,(A \Rightarrow B) \Rightarrow D}{C}\,B, D \Rightarrow C}{E}\,((A \Rightarrow B) \Rightarrow C) \Rightarrow E}{F}\,(B \Rightarrow E) \Rightarrow F \quad .$$

The numbers indicate which assumptions are discharged at the application of an inference. Note that this derivation is purely structural. We are not working in a formal system in which specific primitive rules are available. Conceptually this corresponds, for example, to a derivation of A from A, B, C in standard natural deduction, which can be obtained without any specific primitive rule of inference. In fact, it just consists of the assumption A being derived from itself, notated as

$$A \;,$$

which in our notation corresponds to (5). By incorporating the idea of assuming and applying a rule into the apparatus of deduction, higher-level rules present a much richer structural framework than natural deduction, where one essentially just has the assumption of formulas. If instead we use a sequent-style framework, the structural apparatus is of course more fine-grained due to the availability of structural rules such as thinning and contraction. There, too, rules of higher levels provide additional structural means of expression that go way beyond what is available in the standard context (see Schroeder-Heister, 1987).

The notion of a derivation in a formal system K can now be defined as follows. If a certain set of rules is specified as the set of *primitive rules* of the system K, then a derivation of B from rules $R_1, \ldots, R_n$ in K is a derivation (simpliciter, i.e. in the sense defined in the previous paragraph), such that every rule on which B depends, is either a primitive rule of K or occurs in $R_1, \ldots, R_n$. We say that B is derivable from rules $R_1, \ldots, R_n$ in K, if there is a derivation of B from $R_1, \ldots, R_n$ in K, formally written as $R_1, \ldots, R_n \vdash_K B$. In this case we also say that the rule $R_1, \ldots, R_n \Rightarrow B$ is derivable in K.

As a typical example, consider the system K_{mp} in which every rule of the form $A \to B, A \Rightarrow B$ (i.e., modus ponens) is primitive. Then the higher-level rule $A \to B, ((A \Rightarrow B) \Rightarrow C) \Rightarrow C$ is derivable for every C in K_{mp}, as the following derivation shows:

$$\scriptstyle(1)\ \displaystyle\frac{\dfrac{\dfrac{}{A \to B}\,A \to B \qquad \dfrac{}{A}\,[A]^{(1)}}{B}\,\langle A \to B, A \Rightarrow B\rangle}{C}\,(A \Rightarrow B) \Rightarrow C \qquad .$$

Here, the rule enclosed in angle brackets $\langle \ldots \rangle$ is a primitive rule of K_{mp}. Thus $A \to B, ((A \Rightarrow B) \Rightarrow C) \vdash_{K_{mp}} C$.

Conversely, if for every C the rule $A \to B, ((A \Rightarrow B) \Rightarrow C) \Rightarrow C$ is primitive in a system K_{hl}, then the rule $A \to B, A \Rightarrow B$ is derivable in K_{hl}, i.e., $A \to B, A \vdash_{K_{hl}} B$:

$$\scriptstyle(1)\ \displaystyle\frac{\dfrac{}{A \to B}\,A \to B \qquad \dfrac{\dfrac{}{A}\,A}{B}\,[A \Rightarrow B]^{(1)}}{B}\,\langle A \to B, ((A \Rightarrow B) \Rightarrow B) \Rightarrow B\rangle \quad .$$

This shows that modus ponens and the schema $A \to B, ((A \Rightarrow B) \Rightarrow C) \Rightarrow C$ are equivalent (in the second derivation we have used an instance of the schema with B substituted for C). The content of the latter rule becomes clearer, if we write it in two-dimensional schema notation:

$$\frac{A \to B \qquad \begin{matrix}[A \Rightarrow B]\\ C\end{matrix}}{C} \qquad .$$

This is the generalised higher-level elimination rule for implication which is framed according to the model of $\vee$-elimination and is equivalent to modus ponens.

The structural system with higher-level rules in a language L can be embedded into conjunction-implication logic in the following sense. Let $L_{+(\wedge,\to)}$ be the language resulting from L by adding conjunction and implication as connectives. If L contains already conjunction and implication, then L and $L_{+(\wedge,\to)}$ are identical. We translate higher-level rules R and lists Γ of higher-level rules into $L_{+(\wedge,\to)}$-formulas R^f and Γ^f in the following obvious way:

Definition 2.

- $A^f := A$, *if* A *is a formula.*
- $(R_1, \ldots, R_n \Rightarrow A)^f := R_1^f \wedge \ldots \wedge R_n^f \to A$ *for a rule* $R_1, \ldots, R_n \Rightarrow A$.
- $\Gamma^f := R_1^f \wedge \ldots \wedge R_n^f$, *if* Γ *is the list of rules* $R_1, \ldots, R_n$.

For example, suppose L contains $\bot$, $\vee$, and $\to$. Then the rule $A \vee B \Rightarrow B \vee A$ is translated into the the formula $(A \vee B) \to (B \vee A)$, the rule $A \to \bot, A \vee B \Rightarrow B$ is translated into the formula $((A \to \bot) \wedge (A \vee B)) \to B$, and the rule $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ is translated into the formula $((A \to B) \to A) \to A$.

Let $K_{(\wedge\rightarrow)}$ be the system, which, for formulas A, B in $L_{+(\wedge,\rightarrow)}$, has the standard rules for conjunction and implication as primitive rules:

$$A, B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \quad A \wedge B \Rightarrow B$$

$$(A \Rightarrow B) \Rightarrow A \rightarrow B \qquad A \rightarrow B, A \Rightarrow B \ .$$

Then in $K_{(\wedge\rightarrow)}$ we can show that for any rule R

$$R \dashv\vdash R^f \ , \tag{8}$$

and, more generally, for any list of rules Γ,

$$\Gamma \dashv\vdash \Gamma^f \tag{9}$$

holds, where, as usual, the derivability of a list means the derivability of each of its elements. Thus, technically, the calculus with higher-level rules can be viewed as a notational variant of conjunction-implication logic. However, foundationally, the idea of rules of higher levels is considered the primary concept. In this paper the translation by means of (8) and (9) allows us to use results established for intuitionistic propositional logic as results about the expressive power of higher-level rules.

Conversely, we can embed conjunction-implication logic into the system with higher-level rules over L. We first define a left-iterated conjunction-implication formula:

Definition 3.

- *Every propositional variable p is a left-iterated conjunction-implication formula of degree 0.*
- *If $B_1, \ldots, B_n$ are left-iterated conjunction-implication formulas of maximum degree ℓ, then $(B_1 \wedge \ldots \wedge B_n) \rightarrow p$ is a left-interated conjunction-implication formula of degree $\ell + 1$.*
- *Any conjunction $B_1 \wedge \ldots \wedge B_n$ of left-iterated conjunction-implication formulas $B_1, \ldots, B_n$ of maximum degree ℓ is a left-iterated conjunction-implication formula of degree ℓ.*

Left-iterated conjunction-implication formulas are translated directly into rules and lists of rules:

Definition 4.

- $p^r := p$ *for propositional variables* p
- $((B_1 \wedge \ldots \wedge B_n) \ \rightarrow \ p)^r := B_1^r, \ldots, B_n^r \Rightarrow p$
- $(B_1 \wedge \ldots \wedge B_n)^r := B_1^r, \ldots, B_n^r$.

For left-iterated conjunction-implication formulas C we can, in analogy to (8) and (9), show that

$$C \dashv\vdash C^r$$

holds in $K_{(\wedge\to)}$. Any conjunction-implication formula B, i.e., any formula B only containing conjunction and implication, can be transformed into a (uniquely determined) equivalent left-iterated conjunction-implication formula B' by iterating the following rewrite instructions until an irreducible formula is reached:

- *Replace any subformula of the form* $C \to D_1 \wedge \ldots \wedge D_n$ *with* $(C \to D_1) \wedge \ldots \wedge (C \to D_n)$.
- *Replace any subformula of the form* $C \to (D \to E)$ *with* $(C \wedge D) \to E$.

The *degree* of a conjunction-implication formula B is defined as the degree of the left-iterated conjunction-implication formula B' associated with it. Thus for any conjunction-implication formula B we have

$$B \dashv\vdash B'^r. \tag{10}$$

As (10) is closed under substitution, we can extend this translation to arbitrary substitution instances of conjunction-implication formulas in $L_{+(\wedge,\to)}$. If a formula A of $L_{+(\wedge,\to)}$ is given as a substitution instance $B\sigma$ of a conjunction-implication formula B for a substitution σ, we can translate A into the rule $B'^r\sigma$.

Note that this translation from formulas A to rules $B'^r\sigma$ is not deterministic, as it depends on the choice of the conjunction-implication formula B and the substitution σ. For example, a formula $(A_1 \vee A_2) \to (A_3 \wedge (A_4 \to A_5))$ can be viewed as a substitution instance of the atom p, or of the formula $p \to q$, or of the formula $p \to (p_3 \wedge (p_4 \to p_5))$, etc. Depending on which formula is chosen, a different translation is obtained: In the first case it is translated into itself conceived as a level-0-rule, in the second case it is translated into the rule $A_1 \vee A_2 \Rightarrow A_3 \wedge (A_4 \to A_5)$, in the third case it is translated into the two-element list of rules $(A_1 \vee A_2 \Rightarrow A_3), (A_1 \vee A_2, A_4 \Rightarrow A_5)$.

The transition from rules to formulas is deterministic, since, when we start with rules, *all* rule arrows are replaced with implication signs and *all* commas by conjunction signs. However, when we start with formulas, it is not determined whether an implication or conjunction sign remains part of a formula or becomes a rule arrow or comma. If the language $L_{+(\wedge,\to)}$ does not contain any connective beyond conjunction and implication, we can always choose A'^r to be the unique translation of A into a rule or list of rules. This translation will be used in our definition of harmony and in our Main Theorem in Section 3. It represents the link between our formal exposition in Section 4, which uses the formalism of intuitionistic propositional logic, and our results about the forms and levels of rules. If we define the degree of A to be the (uniquely determined) degree of A', the degree of A is identical to the level of the rule A'^r. If the formulas in a rule R contain implications, then the degree of R^f can be greater than the level of R. In fact, $(R^f)'^r$ is not necessarily identical to R, whereas $(A'^r)^f$ is at least identical to A'. For example, $((p_1 \Rightarrow (p_2 \to p_3))^f)'^r = (p_1 \to (p_2 \to p_3))'^r = (p_1, p_2 \Rightarrow p_3)$, whereas $((p_1 \to (p_2 \to p_3))'^r)^f = (p_1 \wedge p_2) \to p_3 = (p_1 \to (p_2 \to p_3))'$.

3 Harmony and Main Theorem

Various definitions of proof-theoretic harmony exist in the literature. Most definitions start from given introduction rules or (more rarely) from elimination rules and define harmony when the correponding elimination or introduction rules, respectively, relate to the given rules in a certain way. We instead propose to define a notion of harmony which applies to given introduction and elimination rules rather than starting from one of these two kinds of rules. We first define what an introduction and elimination rule should look like and then set up a criterion according to which a given set of introduction rules and a given set of elimination rules for a connective c are in harmony with each other. In our definition of harmony, we do not hesitate to use standard propositional logic. This is not circular as we are not aiming at justifying the rules for the standard connectives, but want to establish a general technical result about the possible forms of introduction and elimination rules. Our approach is therefore 'reductive' rather than 'foundational' in the sense of Schroeder-Heister (2014c). For a foundational approach where harmony is directly defined in terms of rules rather than formulas representing them see Schroeder-Heister (2014a).

As the general schema of an introduction rule for c we propose the following:

$$(c\,\mathrm{I})\quad \frac{\begin{matrix}[\Gamma_1] & & [\Gamma_\ell] \\ B_1 & \ldots & B_\ell\end{matrix}}{c(A_1,\ldots,A_n)}\ , \tag{11}$$

where the Γ_i are (possibly empty) lists of rules, which can be discharged at the application of $(c\,\mathrm{I})$. In the premisess of this schema no schematic letters beyond $A_1,\ldots,A_n$ are allowed to occur. As a limiting case we allow for $\ell = 0$ (which covers the case of the truth constant $\top$). Analogous schemas have been proposed and discussed by von Kutschera (1968), Prawitz (1979), Schroeder-Heister (1984) and Francez and Dyckhoff (2012). We do not consider the case where the Γ_i and B_i contain connectives already defined, as the formal results of our paper concern the relationship between introduction and elimination rules for connectives characterised independently. The choice of this schema for introduction rules is quite plausible: It allows for any list of higher-level rules as conditions for the introduction of c.

Since we assume that conjunction and implication are already available, we may, in view of the fact that rules and lists of rules can be expressed by implications (see (8) and (9)), equivalently replace (11) with the rule

$$\frac{(\Gamma_1^f \to A_1)\wedge\ldots\wedge(\Gamma_\ell^f \to A_\ell)}{c(A_1,\ldots,A_n)}\ . \tag{12}$$

If the level of (11) is $k+1$, then the degree of the premiss of (12) is k. Slightly more generally, in view of the translation from formulas to rules (10), we can assume that an introduction rule of level $k+1$ is propositionally represented by a rule of the form

$$\frac{B}{c(A_1,\ldots,A_n)} \tag{13}$$

where B is a conjunction-implication formula of degree k in which no schematic letters beyond $A_1,\ldots,A_n$ occur.

As the general schema of an elimination rule for c we take the following:

$$(c\,\mathrm{E}) \quad \frac{c(A_1,\ldots,A_n) \quad \begin{matrix}[\Gamma_1]\\ B_1\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma_\ell]\\ B_\ell\end{matrix}}{C}\ , \tag{14}$$

where the Γ_i are (possibly empty) lists of rules. $c(A_1,\ldots,A_n)$ is called the *major premiss* of $(c\,\mathrm{E})$, and the remaining premisses are called the *minor premisses* of $(c\,\mathrm{E})$. We allow for $\ell = 0$, in which case minor premisses are lacking. We do not impose any restrictions on the schematic letters occurring in $(c\,\mathrm{E})$. They may (and will normally) comprise $A_1,\ldots,A_n$, but any number of schematic letters beyond $A_1,\ldots,A_n$ may be present. This generalises the fact that in elimination rules such as $\vee$-elimination

$$\frac{A_1 \vee A_2 \quad \begin{matrix}[A_1]\\ C\end{matrix} \quad \begin{matrix}[A_2]\\ C\end{matrix}}{C}$$

the additional schematic letter C is used as minor premiss and conclusion. The choice of (14) as elimination schema is quite plausible: We should be able to choose anything whatsoever as possible consequence of $c(A_1,\ldots,A_n)$, which means that the minor premisses and the conclusion should not be constrained in any way.

Using the propositional translation of rules (see (8),(9)), we can translate (14) as follows:

$$\frac{c(A_1,\ldots,A_n) \qquad (\Gamma_1^f \to B_1) \wedge \ \ldots \ \wedge (\Gamma_\ell^f \to B_\ell)}{C} \tag{15}$$

More generally, we can propositionally represent an elimination rule (14) of level $d+1$ by

$$\frac{c(A_1,\ldots,A_n) \quad B}{C}$$

or

$$\frac{c(A_1,\ldots,A_n)}{B \to C} \tag{16}$$

where B is a left-iterated conjunction-implication formula of degree d and where, as a limiting case, B can be lacking, in which case $B \to C$ is just C.

Suppose for c a list $c\mathcal{I}$ of introduction rules of the form (11) and a list $c\mathcal{E}$ of elimination rules of the form (14) are given. Passing to their propositional representations (13) and (16), we suppose that m introduction rules

$$\frac{B_1}{c(A_1,\ldots,A_n)} \quad \ldots \quad \frac{B_m}{c(A_1,\ldots,A_n)} \tag{17}$$

and k elimination rules

$$\frac{c(A_1,\ldots,A_n)}{D_1\to C_1} \quad \ldots \quad \frac{c(A_1,\ldots,A_n)}{D_k\to C_k} \tag{18}$$

are given. Since introduction rules are understood disjunctively, we may compress (17) to

$$\frac{B_1\vee\ldots\vee B_m}{c(A_1,\ldots,A_n)}\ . \tag{19}$$

The disjunction $B_1\vee\ldots\vee B_m$ is called the *introduction meaning* c^I of c. Since elimination rules are understood conjunctively, we may compress (18) to

$$\frac{c(A_1,\ldots,A_n)}{(D_1\to C_1)\wedge\ldots\wedge(D_k\to C_k)}\ .$$

As these schematic letters occur only in the conclusion, we may equivalently write

$$\frac{c(A_1,\ldots,A_n)}{\forall\!\forall((D_1\to C_1)\wedge\ldots\wedge(D_k\to C_k))} \tag{20}$$

where the quantifier $\forall\!\forall$ is understood as binding all schematic letters in $(D_1\to C_1)\wedge\ldots\wedge(D_k\to C_k)$ except $A_1,\ldots,A_n$.[4] The formula $\forall\!\forall((D_1\to C_1)\wedge\ldots\wedge(D_k\to C_k))$ is called the *elimination meaning* c^E of c. For example, in the case of disjunction, this rule becomes

$$\frac{A\vee B}{\forall C(((A\to C)\wedge(B\to C))\to C)}$$

with $\forall C(((A\to C)\wedge(B\to C))\to C)$ being the elimination meaning of disjunction. If we consider a biimplication $\equiv$ with the general elimination rules

$$\frac{A\equiv B \quad A \quad \begin{matrix}[B]\\ C\end{matrix}}{C} \qquad \frac{A\equiv B \quad B \quad \begin{matrix}[A]\\ C\end{matrix}}{C}$$

then (20) takes the form

$$\frac{A\equiv B}{\forall C(((A\wedge(B\to C))\to C)\wedge((B\wedge(A\to C))\to C))}$$

[4]Note that for simplicity we here use schematic letters as variables over which we can quantify. If we wanted to be absolutely precise, we should use propositional variables for that purpose.

with $\forall C(((A\wedge(B\to C))\to C)\wedge((B\wedge(A\to C))\to C))$ being the elimination meaning of $\equiv$. For further examples see Schroeder-Heister (2014c).

Now we can define our notion of harmony in terms of the propositional representations of the introduction and elimination rules for c. We simply say that proposed introduction and elimination rules for c are in *harmony* with each other, if introduction and elimination meaning of c according to these rules match, i.e., if c^I and c^E are equivalent in second order intuitionistic propositional logic.

Definition 5.

Suppose a list $c\mathcal{I}$ *of* m *introduction rules for* c *of the form* (11)*, and a list* $c\mathcal{E}$ *of* k *elimination rules for* c *of the form* (14) *are given. Suppose their propositional representations are* (17) *and* (18)*. Then* $B_1\vee\ldots\vee B_m$ *is called the* introduction meaning c^I *of* c*, and* $\forall\forall((D_1\to C_1)\wedge\ldots\wedge(D_k\to C_k))$ *is called the* elimination meaning c^E *of* c*.*

$c\mathcal{I}$ *and* $c\mathcal{E}$ *are in* harmony *with each other, if introduction meaning* c^I *and elimination meaning* c^E *match, i.e., if*

$$c^I \dashv\vdash c^E \text{ , i.e., } \; B_1\vee\ldots\vee B_m \dashv\vdash \forall\forall((D_1\to C_1)\wedge\ldots\wedge(D_k\to C_k))$$

where $\dashv\vdash$ *denotes interderivability in second-order intuitionistic propositional logic.*

If $c\mathcal{I}$ and $c\mathcal{E}$ are in harmony with each other, then from (19) and (20) it follows immediately that $c(A_1,\ldots,A_n)\dashv\vdash c^I$ as well as $c(A_1,\ldots,A_n)\dashv\vdash c^E$, where $\dashv\vdash$ denotes interderivability in second-order intuitionistic propositional logic extended with introduction and elimination rules for c. This means that c is explicitly definable in this logic both by c^I and by c^E. For a detailed discussion of this and other features of our notion of harmony see Schroeder-Heister (2014c). For a corresponding notion of harmony which does not rely on second-order propositional logic but uses quantified rules instead, see Schroeder-Heister (2014a).

Using this definition we will show that given introduction rules of maximum level $d+1$, i.e., with the formulas B_i in (17) of maximum degree d, there are not always elimination rules of level $d+1$ or below, i.e. with the formulas D_i in (18) of level d or below. For each d we will give a connective $\star_d$ with two introduction rules in such a way that B_1 is of degree d and B_2 of degree 0. We then show that for any matching $\forall\forall((D_1\to C_1)\wedge\ldots\wedge(D_k\to C_k))$, the formula $(D_1\to C_1)\wedge\ldots\wedge(D_k\to C_k)$ must at least be of degree $d+2$, i.e. some D_i must at least be of degree $d+1$. This means that a level increase when passing from introductions to harmonious eliminations cannot be avoided in this case. Conversely, for each d we will give a connective $\circ_d$ with a single elimination rule of level $d+1$, i.e. with the formula D_1 in (18) of level d, such that for any matching $B_1\vee\ldots\vee B_m$, at least one formula B_i must at least be of degree $d+1$. This means that a level increase when passing from eliminations to harmonious introductions cannot be avoided in this case.

Thus we can formulate our central result.

Main Theorem

(i) For every d, there is a connective $\star_d$ characterised by a set of introduction rules such that the following holds: The introduction rules for $\star_d$ are of maximum level $d+1$, but every set of elimination rules which is in harmony with the given set of introduction rules, contains at least one rule of level greater than $d+1$.

(ii) For every d, there is a connective $\circ_d$ characterised by a singleton set of elimination rules such that the following holds: The elimination rule for $\circ_d$ is of level $d+1$, but every set of introduction rules which is in harmony with the given elimination rule contains at least one rule of level greater than $d+1$.

Proof. We show that the theorem can be reduced to two theorems about formulas in intuitionistic propositional logic, which do not mention rules. These two theorems, which correspond to (i) and (ii), will be proved in the next section. A disjunction $B_1 \vee \ldots \vee B_m$ of conjunction-implication formulas is called a *d-introduction formula*, if the disjuncts are of maximum degree d. In view of the propositional representation (13) of introduction rules a d-introduction formula corresponds to a set $c\mathcal{I}$ of introduction rules for a connective, whose maximum level is $d+1$. A formula of the form $\forall\!\!\forall B$, where $\forall\!\!\forall$ binds all variables beyond $p_1, \ldots, p_{d+1}$ is called a *d-elimination formula*, if B is of degree d. In view of the propositional representation (16) of elimination rules a d-elimination formula corresponds to a set $c\mathcal{E}$ of elimination rules for an n-place connective, whose maximum level is 1 if d is 0, and d if $d > 0$. (An elimination rule can never be of level 0, as it has at least one premiss, namely the major premiss.)

(i) According to Theorem 1 of the next section, for every $d \geq 0$ there is a d-introduction formula F_d, which is not intuitionistically equivalent to any $(d+1)$-elimination formula. The formula F_d has the form $(p_1 \to \ldots \to p_{d+1}) \vee p_{d+2}$, where the implications are bracketed to the left, i.e. F_0 is $p_1 \vee p_2$, F_1 is $(p_1 \to p_2) \vee p_3$, F_2 is $((p_1 \to p_2) \to p_3) \vee p_4$, etc. This means, there is a corresponding connective $\star_d$ with introduction rules of maximum level $d+1$, for which there is no set of harmonious elimination rules of level $d+1$ or below. The connective $\star_d$ is $(d+2)$-ary, with the two introduction rules

$$\frac{\begin{array}{c}[(\ldots A_1 \Rightarrow \ \ldots) \Rightarrow A_d]\\ A_{d+1}\end{array}}{\star_d(A_1, \ldots, A_{d+2})} \qquad \frac{A_{d+2}}{\star_d(A_1, \ldots, A_{d+2})}\ .$$

Hence $\star_0$ has the introduction rules

$$\frac{A_1}{\star_0(A_1, A_2)} \qquad \frac{A_2}{\star_0(A_1, A_2)}$$

and is thus equivalent to disjunction, $\star_1$ has the introduction rules

$$\frac{\begin{array}{c}[A_1]\\ A_2\end{array}}{\star_1(A_1, A_2, A_3)} \qquad \frac{A_3}{\star_1(A_1, A_2, A_3)}$$

and is thus the three-place connective $\star$, for which we demonstrated the non-flattening result in Olkhovikov and Schroeder-Heister (2014), $\star_2$ has the introduction rules

$$\frac{\begin{matrix}[A_1 \Rightarrow A_2]\\ A_3\end{matrix}}{\star_2(A_1, A_2, A_3, A_4)} \qquad \frac{A_4}{\star_2(A_1, A_2, A_3, A_4)}\ ,$$

$\star_3$ has the introduction rules

$$\frac{\begin{matrix}[(A_1 \Rightarrow A_2) \Rightarrow A_3]\\ A_4\end{matrix}}{\star_3(A_1, A_2, A_3, A_4, A_5)} \qquad \frac{A_5}{\star_3(A_1, A_2, A_3, A_4, A_5)}\ ,$$

etc.

(ii) According to Theorem 2 of the next section, for every $d \geq 0$ there is a $(d+1)$-elimination formula G_d, which is not intuitionistically equivalent to any d-introduction formula. The formula G_d has the form $(p_1 \to \ldots \to p_{d+2})$, where the implications are bracketed to the left, i.e. G_0 is $p_1 \to p_2$, G_1 is $(p_1 \to p_2) \to p_3$, G_2 is $((p_1 \to p_2) \to p_3) \to p_4$, etc. This means, there is a corresponding connective $\circ_d$ with an elimination rule of level $d+1$, for which there is no set of harmonious introduction rules of level $d+1$ or below. The connective $\circ_d$ is $(d+2)$-ary, with the elimination rule

$$\frac{\circ_d(A_1, \ldots, A_{d+2}) \qquad \begin{matrix}[(\ldots A_1 \Rightarrow \ \ldots) \Rightarrow A_d]\\ A_{d+1}\end{matrix}}{A_{d+2}}\ .$$

Hence $\circ_0$ has the elimination rule

$$\frac{\circ_0(A_1, A_2) \quad A_1}{A_2}$$

and is thus equivalent to implication, $\circ_1$ has the elimination rule

$$\frac{\circ_1(A_1, A_2, A_3) \quad \begin{matrix}[A_1]\\ A_2\end{matrix}}{A_3}$$

and is thus the three-place connective $\circ$, for which we demonstrated the non-flattening result in Olkhovikov and Schroeder-Heister (2014), $\circ_2$ has the elimination rule

$$\frac{\circ_d(A_1, A_2, A_3, A_4) \qquad \begin{matrix}[A_1 \Rightarrow A_2]\\ A_3\end{matrix}}{A_4}\ ,$$

$\circ_3$ has the elimination rule

$$\frac{\circ_d(A_1, A_2, A_3, A_4, A_5) \qquad \begin{matrix}[(A_1 \Rightarrow A_2) \Rightarrow A_3]\\ A_4\end{matrix}}{A_5}\ ,$$

etc. □

The following section is devoted to the proofs of Theorem 1 and Theorem 2. It pertains to the background of higher-level rules in the way just described, and is, of course, inspired by it. However, the results proved are formally independent of this background and deal with the expressive power of conjunction-implication formulas in (second-order) intuitionistic propositional logic.

4 Theorems on definability by n-elimination and n-introduction formulas

We assume the language L of intuitionistic propositional logic based on a countable set Var of propositional variables and the set $\{\wedge, \vee, \to, \bot\}$ as the set of basic connectives. $L_{(\wedge,\to)}$ and $L_{\to}$ stand for the fragments of this language in their respective restricted sets of connectives. Unlike the languages considered in the previous section, our language here does not contain additional logical constants beyond the four basic connectives. Later, in the definition of n-elimination formulas and the proof of Theorem 1, we will also consider universal propositional quantification.

Let us introduce some notation. We will use notation $A_1 \to \ldots \to A_n$ for the chain of implications assuming that the parentheses are grouped to the right, i. e. that, for instance, $A_1 \to A_2 \to A_3$ stands for $A_1 \to (A_2 \to A_3)$. Every $A \in L_{\to}$ is of the form $B_1 \to \ldots \to B_n \to p$, for the unique $B_1, \ldots, B_n \in L_{\to}$ and $p \in Var$. In this case we will call p the consequent of A and write $p = Con(A)$; we will also call $\{B_1, \ldots, B_n\}$ the set of antecedents of A and write $Ant(A) = \{B_1, \ldots, B_n\}$. We assume $Ant(p) = \varnothing$ for $p \in Var$ so that Con and Ant are defined for every $A \in L_{\to}$. This means that sometimes we will write e. g. $A \to B$ allowing that A is empty, that is to say, that $A \to B = B$.

Next we define the degree $d(A)$ for a formula $A \in L_{\to}$. We do this by the following induction on the complexity of A:

$$d(A) = 0, \text{ if } Ant(A) = \varnothing;$$

$$d(A) = max(\{d(B) \mid B \in Ant(A)\}) + 1 \text{ otherwise.}$$

This definition adapts our earlier definition of a formula degree to the setting of $L_{\to}$. It is easy to see that $d(B_1 \to \ldots \to B_n \to p)$ is equal to the degree of $(B_1 \wedge \ldots \wedge B_n) \to p$ according to Definition 3.

We let K (possibly with various subscripts and/or primes) stand for a finite set of formulas. One can extend the notion of degree onto the finite sets of formulas in $L_{\to}$ in the following natural way:

$$d(K) = max(\{d(A) \mid A \in K\}) \tag{21}$$

We also let $K|_i$ stand for the restriction of K to the set of formulas not exceeding the given degree i. That is to say, we define:

$$K|_i = \{A \in K \mid d(A) \leq i\}.$$

We denote the set of subformulas of a formula A by $Sub(A)$ and we extend this notation onto the finite sets of formulas in the following way:

$$Sub(K) = \bigcup\{Sub(A) \mid A \in K\}.$$

Finally, for $\{A_1, \ldots, A_n\} \in L_\to$ and $p \in Var$ we set

$$Imp(\{A_1, \ldots, A_n\}, p) := \{A_{\pi(1)} \to \ldots \to A_{\pi(n)} \to p \mid \ \pi - \text{permutation on } \{1, \ldots, n\}\}.$$

Of course, all the formulas in $Imp(\{A_1, \ldots, A_n\}, p)$ are intuitionistically equivalent.

Lemma 1. *For every $A \in L_{(\wedge,\to)}$ there are formulas $B_1, \ldots, B_n \in L_\to$ such that A is intuitionistically equivalent to $B_1 \wedge \ldots \wedge B_n$.*

Proof. We prove the lemma by induction on the number k of logical connectives in A.

Basis. If $k = 0$, then $A = p$ for some $p \in Var$, therefore, we set $n := 1$ and $B_1 := p = A$.

Induction step. If $k = m + 1$, then we consider two cases:

Case 1. $A = A_0 \wedge A_1$. By induction hypothesis, there are $C_1, \ldots, C_r, D_1, \ldots, D_s \in L_\to$ such that the following biconditionals are intuitionistically valid:

$$A_0 \leftrightarrow C_1 \wedge \ldots \wedge C_r \tag{22}$$

$$A_1 \leftrightarrow D_1 \wedge \ldots, \wedge D_s \tag{23}$$

Then, of course, A is intuitionistically equivalent to

$$C_1 \wedge \ldots \wedge C_r \wedge D_1 \wedge \ldots, \wedge D_s$$

and we are done.

Case 1. $A = A_0 \to A_1$. Again, applying induction hypothesis, we get that A is intuitionistically equivalent to

$$(C_1 \wedge \ldots \wedge C_r) \to (D_1 \wedge \ldots, \wedge D_s)$$

for appropriate $C_1, \ldots, C_r, D_1, \ldots, D_s \in L_\to$. Then we get the following chain of intuitionistically valid biconditionals:

$$\begin{aligned}
A &\leftrightarrow (C_1 \wedge \ldots \wedge C_r) \to (D_1 \wedge \ldots, \wedge D_s)\\
&\leftrightarrow ((C_1 \wedge \ldots \wedge C_r) \to D_1) \wedge \ldots \wedge ((C_1 \wedge \ldots \wedge C_r) \to D_s)\\
&\leftrightarrow (C_1 \to \ldots \to C_r \to D_1) \wedge \ldots \wedge (C_1 \to \ldots \to C_r \to D_s)
\end{aligned}$$

Since the last formula in this chain is a conjunction of formulas in $L_\to$, we are done. $\square$

If A is intuitionistically valid, we will write $\models A$. Lemma 1 shows that one can extend the notion of degree onto $L_{(\wedge,\to)}$. However, it is not quite unproblematic to define e. g. that

$$d(A) = max(\{d(B_1), \ldots, d(B_n)\}) \mid B_1, \ldots, B_n \in L_{\to}, \text{ and } \models A \leftrightarrow (B_1 \wedge \ldots \wedge B_n)\}),$$

because, for instance $p \wedge q$ is intuitionistically equivalent to both itself and $((p \to p) \to p) \wedge q$ and the corresponding subsets of $L_{\to}$, $\{p, q\}$ and $\{(p \to p) \to p, q\}$ have different degrees. Luckily enough, the proof of our Lemma 1 actually yields a deterministic algorithm which calculates for a given $A \in L_{(\wedge,\to)}$ exactly one set of conjuncts $\{B_1^A, \ldots, B_n^A\}$. This algorithm is conservative over $L_{\to}$ in the sense that it always calculates the set $\{A\}$ if $A \in L_{\to}$. So we can assume the following definition for arbitrary $A \in L_{(\wedge,\to)}$

$$d(A) = d(\{B_1^A, \ldots, B_n^A\}).$$

Of, course, we can go one more step further, and assume the definition (21) for arbitrary finite $K \subseteq L_{(\wedge,\to)}$.

For a finite $K \subseteq L_{\to}$ and natural i, j we define $S_j^i(K)$ in the following way.

$$S_j^0(K) = K|_0 \text{ for every } j. \ ;$$

$$S_0^{i+1}(K) = K|_{i+1};$$

$$\begin{aligned} S_{j+1}^{i+1}(K) = {} & S_j^{i+1}(K) \cup \\ & \cup \{D \to B \in Sub(K)|_{i+1} \mid Con(B) \in S^i(S_j^{i+1}(K)|_i \cup Ant(B))\} \cup \\ & \cup \{D \to B \in Sub(K)|_{i+1} \mid \exists \Gamma \subseteq S_j^{i+1}(K) \exists C \in Sub(K)|_{i+1} \\ & \qquad (C \to C \in Imp(\Gamma \cup Ant(B), Con(B)))\}; \end{aligned}$$

$$S^{i+1}(K) = \bigcup_j (S_j^{i+1}(K)).$$

$$S(K) = \bigcup_i S^i(K).$$

In the above defintions, D is allowed to be empty, that is to say, we might have $D \to B = B$. Thus, e. g. for arbitrary finite $K \subseteq L_{\to}$ and natural j:

$$S^0(K) = S_j^0(K) = K|_0.$$

We establish some further quick facts about this new notion:

Lemma 2. *All of the following are true about arbitrary finite $K \subseteq L_{\to}$:*

(a) For every natural i, $S^i(K) \subseteq Sub(K)|_i$ and hence is finite. Therefore, $S(K) \subseteq Sub(K)$ and is finite.

(b) If $K \subseteq K'$, then for every natural i, $S^i(K) \subseteq S^i(K')$.

(c) For all i, j and k, if $i \leq j$, then $S^k_i(K) \subseteq S^k_j(K)$.

(d) If $i \leq j$, then for any natural k, $S^i_k(K) \subseteq S^j_k(K)$; in particular, $S^i(K) \subseteq S^j(K)$.

(e) For arbitrary $K' \subseteq S(K)$ there are some natural i, j, such that $K' \subseteq S^i_j(K)$.

Proof. (a) and (b) are immediate from the definition, and (c) follows by an obvious induction.

We show (d) by induction on j.

Basis-1. For $j = 0$ we have $i = 0 = j$ so $S^i_k(K) \subseteq S^j_k(K)$ is immediate for arbitrary K.

Induction hypothesis-1. Assume that for all K and for all $j \leq r$ it is true that if $i \leq j$, then for any natural k, $S^i_k(K) \subseteq S^j_k(K)$; in particular, $S^i(K) \subseteq S^j(K)$.

Induction step-1. Let $j = r + 1$ and choose some finite $K \subseteq L_{\to}$. We show that $S^i_k(K) \subseteq S^j_k(K)$ by (another) induction on k. If $i = 0$ then clearly

$$S^0_k(K) = K|_0 \subseteq K|_j = S^j_0(K) \subseteq S^j_k(K),$$

for arbitrary natural k and we are done. So we will be assuming that $i = t+1$ for some natural t.

Basis-2. For $k = 0$ we have the inclusion

$$S^i_0(K) = K|_i \subseteq K|_j = S^j_0(K)$$

by the assumption that $i \leq j$ and the definition of $K|_n$.

Induction hypothesis-2. Assume that for $k \leq s$ it is true that $S^i_k(K) \subseteq S^j_k(K)$.

Induction step-2. Let $k = s + 1$. Now, assume that $A \in S^i_k(K) = S^{t+1}_{s+1}(K)$. We will show that $A \in S^j_k(K)$. Three cases are possible:

Case 1. $A \in S^{t+1}_s(K)$. Then, by IH-2, $A \in S^{r+1}_s(K) \subseteq S^{r+1}_{s+1}(K) = S^j_k(K)$ and we are done.

Case 2. There is $D \to B \in Sub(K)|_{t+1}$ such that $A = D \to B$ and $Con(B) \in S^t(S^{t+1}_s(K)|_t \cup Ant(B))$. Then, of course, $A = D \to B \in Sub(K)|_{r+1}$ and we also have the following inclusions:

$$S^{t+1}_s(K) \subseteq S^{r+1}_s(K) \tag{24}$$

(by IH-2)

$$S^t(S^{t+1}_s(K)|_t \cup Ant(B)) \subseteq S^t(S^{r+1}_s(K)|_t \cup Ant(B)) \tag{25}$$

(from (24) by (b))

$$t \leq r \tag{26}$$

(from $t + 1 = i \leq j = r + 1$)

$$S^t(S^{r+1}_s(K)|_t \cup Ant(B)) \subseteq S^r(S^{r+1}_s(K)|_t \cup Ant(B)) \tag{27}$$

(from (25), (26) by IH-1)

$$S^{r+1}_s(K)|_t \subseteq S^{r+1}_s(K)|_r \tag{28}$$
(from (26))
$$S^r(S^{r+1}_s(K)|_t \cup Ant(B)) \subseteq S^r(S^{r+1}_s(K)|_r \cup Ant(B)) \tag{29}$$
(from (28) by (b))
$$S^t(S^{t+1}_s(K)|_t \cup Ant(B)) \subseteq S^r(S^{r+1}_s(K)|_r \cup Ant(B)) \tag{30}$$
(from (25), (27), and (29))

The inclusion (30) shows that $Con(B) \in S^r(S^{r+1}_s(K)|_r \cup Ant(B))$, which, in turn, means that $A \in S^{r+1}_{s+1}(K) = S^j_k(K)$ and we are done.

Case 3. There is $D \to B \in Sub(K)|_{t+1}$ such that $A = D \to B$ and

$$\exists\Gamma \subseteq S^{t+1}_s(K)\exists C \in Sub(K)|_{t+1}(C \to C \in Imp(\Gamma \cup Ant(B), Con(B))).$$

Choose some appropriate Γ and C. Then, of course, $\Gamma \subseteq S^{t+1}_s(K) \subseteq S^{r+1}_s(K)$ by IH-2 and $C \in Sub(K)|_{t+1} \subseteq Sub(K)|_{r+1}$ by assumption that $t+1 = i \leq j = r+1$. Therefore, we have

$$\exists\Gamma \subseteq S^{r+1}_s(K)\exists C \in Sub(K)|_{r+1}(C \to C \in Imp(\Gamma \cup Ant(B), Con(B))),$$

which means that $A = D \to B \in S^{r+1}_{s+1}(K) = S^j_k(K)$ and we are done.

Finally, we need to prove (e). Assume that $K' \subseteq S(K)$. Then, by (a), K' is a finite subset of $L_\to$, say, $K' = \{A_1, \ldots, A_n\}$. Then, by definition of $S(K)$, one can choose natural $i_1, \ldots, i_n$ such that for every $1 \leq t \leq n$ it is true that $A_{i_t} \in S^{i_t}(K)$. Then set $i := max(\{i_1, \ldots, i_n\})$. It follows from (d) that $K' \subseteq S^i(K)$. Therefore, one can also choose natural $j_1, \ldots, j_n$ such that for every $1 \leq t \leq n$ it is true that $A_{j_t} \in S^i_{j_t}(K)$. Again, we set $j := max(\{j_1, \ldots, j_n\})$ and it follows by (c) that $K' \subseteq S^i_j(K)$. □

It follows from Lemma 2 that for every finite K the hierarchy of sets $S^i_j(K)$ has a fixpoint. More precisely, we have the following corollary:

Corollary 1. *For arbitrary finite $K \subseteq L_\to$ there are some natural i, j such that*

$$S(K) = S^i_j(K) = S^i(K).$$

Proof. By Lemma 2 (a), $S(K)$ is its own finite subset, hence by Lemma 2 (e) $S(K) \subseteq S^i_j(K)$ for some i, j. We now have the following set of inclusions:

$$S(K) \subseteq S^i_j(K) \subseteq S^i(K) \subseteq S(K),$$

which completes the proof. □

Further, we will need the following constructions on the set of intuitionistic models:

1. If $\mathcal{M} = \langle W, R, V\rangle$ is an intuitionistic model and $K \subseteq Var$, then $\mathcal{M} + K = \langle W, R, V'\rangle$, where for every $p \in Var$ and every $w \in W$

   $$p \in V'(w) \Leftrightarrow (p \in V(w) \vee p \in K).$$

2. If for every $1 \leq i \leq n$, $\mathcal{M}_i = \langle W_i, R_i, V_i\rangle$ is an intuitionistic model, $W_i \cap W_j = \varnothing$ for arbitrary $1 \leq i < j \leq n$, and $w \notin \bigcup_1^n W_i$, then $\Sigma(w, \mathcal{M}_1, \ldots, \mathcal{M}_n) = \langle W, R, V\rangle$ is as follows:

   (a) $W = \bigcup_1^n W_i \cup \{w\}$.

   (b) $R = \bigcup_1^n R_i \cup \{\langle w, v\rangle \mid v \in W\}$.

   (c) $V = \bigcup_1^n V_i$.

Lemma 3. *For arbitrary $A \in L_{\to}$ and finite $K \subseteq L_{\to}$:*

If $A \in S(K)$, then $K \models A$.

Proof. Indeed, if $A \in S(K)$, then $A \in S^i(K)$ for some natural i. We will show that in this case $K \models A$ by induction on i.

Basis-1. Let $i = 0$. Then for an arbitrary K, if $A \in S^0(K) = K|_0$ then, of course, $K \models A$.

Induction hypothesis-1. Assume that for all K and for all $i \leq k$ it is true that if $A \in S^i(K)$, then $K \models A$.

Induction step-1. Let $i = k + 1$, and choose some finite $K \subseteq L_{\to}$. We will show that if $A \in S_j^{k+1}(K)$, then for an arbitrary K, we have $K \models A$ by induction on j.

Basis-2. Let $j = 0$. Then $A \in S_0^{k+1}(K) = K|_{k+1}$ and of course $K \models A$.

Induction hypothesis-2. Assume that for $j \leq m$ it is true that if $A \in S_j^{k+1}(K)$, then $K \models A$.

Induction step-2. Let $j = m + 1$ and let $A \in S_{m+1}^{k+1}(K)$. Then three cases are possible:

Case 1. If $A \in S_m^{k+1}(K)$, then we are done by IH-2.

Case 2. Let $A = D \to B \in Sub(K)|_{k+1}$ and

$$Con(B) \in S^k(S_m^{k+1}(K)|_k \cup Ant(B)).$$

Now, by IH-1 we have that

$$S_m^{k+1}(K)|_k \cup Ant(B) \models Con(B),$$

whence by deduction theorem we get

$$S_m^{k+1}(K)|_k \models \bigwedge(Ant(B)) \to Con(B) \tag{31}$$

and further

$$S_m^{k+1}(K)|_k \models B.$$

On the other hand, we know by IH-2 that for every $C \in S_m^{k+1}(K)$ (hence for every $C \in S_m^{k+1}(K)|_k$) we have

$$K \models C. \tag{32}$$

From (31) and (32) we get

$$K \models B,$$

and, further,

$$K \models D \to B = A.$$

Case 3. Let $A = D \to B \in Sub(K)|_{k+1}$ and

$$\exists \Gamma \subseteq S_m^{k+1}(K) \exists C \in Sub(K)|_{k+1} (C \to C \in Imp(\Gamma \cup Ant(B), Con(B))).$$

Then, since we of course have that

$$\models C \to C,$$

we must, by deduction theorem, also have that

$$\Gamma \cup Ant(B) \models Con(B),$$

whence, again by deduction theorem, we get

$$\Gamma \models \bigwedge(Ant(B)) \to Con(B),$$

whence, further

$$\Gamma \models B$$

and

$$\Gamma \models D \to B = A.$$

Since we have, by IH-2, that

$$\forall E \in \Gamma (K \models E),$$

we finally get that

$$K \models A,$$

which completes the proof. $\square$

Lemma 4. *Let $K \subseteq L_{\to}$. Then there is an intuitionistic model $\mathcal{M}(K)$ such that $A \in Sub(K)$ is forced by the root of this model iff $A \in S^{d(K)}(K)$.*[5]

Proof. By induction on $d(K)$.

Basis-1. Let $d(K) = 0$. Then, for an arbitrary K, set $\mathcal{M}(K) := \langle \{w\}, \{\langle w, w\rangle\}, V\rangle$, where

$$\forall p \in Var(w \in V(p) \Leftrightarrow p \in K).$$

Induction hypothesis-1. Assume that for every $K \subseteq L_{\to}$ such that $d(K) \leq i$ there is an intuitionistic model $\mathcal{M}(K)$ such that $A \in Sub(K)$ is forced by the root of this model iff $A \in S^{d(K)}(K)$.

Induction step-1. Let $d(K) = i + 1$. Then consider the set $Sub(K)$. It can be partitioned into three subsets as follows:

$$S_1 = \{A \in Sub(K) \mid A \in S^{i+1}(K) \wedge Con(A) \in S^{i+1}(K)|_0\};$$

$$S_2 = \{A \in Sub(K) \mid A \in S^{i+1}(K) \wedge Con(A) \notin S^{i+1}(K)|_0\};$$

$$S_3 = \{A \in Sub(K) \mid A \notin S^{i+1}(K)\}.$$

Since $S_3 \subseteq Sub(K)$, it is finite, say $S_3 = \{A_1, \dots, A_n\}$. Let $1 \leq m \leq n$. We know that

$$d(Ant(A_m)) < d(A_m) \leq d(K) = i + 1.$$

Therefore, $d(Ant(A_m) \cup S^{d(K)}(K)|_i) \leq i$. It follows by IH-1 that there are intuitionistic models

$$\mathcal{M}(Ant(A_1) \cup S^{i+1}(K)|_i), \dots, \mathcal{M}(Ant(A_n) \cup S^{i+1}(K)|_i)$$

such that for every $1 \leq m \leq n$ it is true that, if w_m is the root of $\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i)$, then for every $B \in Sub(Ant(A_m) \cup S^{i+1}(K)|_i)$:

$$\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i), w_m \Vdash B \Leftrightarrow B \in S^i(Ant(A_m) \cup S^{i+1}(K)|_i).$$

We now choose w which is not present in the set of worlds of $\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i)$ for any $1 \leq m \leq n$, and set:

$$\mathcal{M}(K) = \Sigma(w, \mathcal{M}(Ant(A_1) \cup S^{i+1}(K)|_i), \dots, \mathcal{M}(Ant(A_n) \cup S^{i+1}(K)|_i)) + S^{i+1}(K)|_0.$$

Note that this construction 'changes nothing' for the worlds in the submodels of the form $\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i)$ in the sense that for an arbitrary $1 \leq m \leq n$ and

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[5] We can even assume, by the method of the proof given below, that the height of $\mathcal{M}(K)$ does not exceed $d(K)$, although this particular fact is not relevant to our main result.

arbitrary world $v \in \mathcal{M}(K)$, if $v \in \mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i)$, then for every intuitionistic propositional formula A it is true that

$$\mathcal{M}(K), v \Vdash A \Leftrightarrow \mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i), v \Vdash A.$$

This follows from the definition of the above constructions and the fact that for every $1 \leq m \leq n$ we must have

$$\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i), v \Vdash S^{i+1}(K)|_i \supseteq S^{i+1}(K)|_0,$$

so the addition of $S^{i+1}(K)|_0$ can only make a difference in the root of $\mathcal{M}(K)$.

Let us verify that

$$\forall(B \in Sub(K))(\mathcal{M}(K), w \Vdash B \Leftrightarrow B \in S^{i+1}(K))$$

by induction on $d(B) \leq i + 1$.

Basis-2. If $d(B) = 0$ then we have

$$B \in S^{i+1}(K) \Leftrightarrow B \in S^{i+1}(K)|_0 \Leftrightarrow \mathcal{M}(K), w \Vdash B,$$

by definition of $\mathcal{M}(K)$.

Induction hypothesis-2. Assume that for every $B \in Sub(K)$, if $d(B) \leq j < i + 1$, then

$$\mathcal{M}(K), w \Vdash B \Leftrightarrow B \in S^{i+1}(K).$$

Induction step-2. Let $d(B) = j + 1$. Three cases are possible here:

Case 1. $B \in S_1$. Then $Con(B) \in S^{i+1}(K)|_0$, which, by definition of $\mathcal{M}(K)$, means that $\mathcal{M}(K), w \Vdash B$.

Case 2. $B \in S_2$. Then $Con(B) \notin S^{i+1}(K)|_0$, which, by definition of $\mathcal{M}(K)$, means that $\mathcal{M}(K), w \nVdash Con(B)$. On the other hand, $\mathcal{M}(K), w \nVdash Ant(B)$. For assume otherwise. Since $d(B) = j + 1$, we have $d(Ant(B)) \leq j$, therefore, it follows by IH-2 from $\mathcal{M}(K), w \Vdash Ant(B)$ that $Ant(B) \subseteq S^{i+1}(K)|_j \subseteq S^{i+1}(K)$. Since we know that $S^{i+1}(K)$ is finite, we can choose some natural u, for which $S^{i+1}(K) = S^{i+1}_u(K)$. Then we have $Ant(B) \subseteq S^{i+1}_u(K)$. Also, we have $B \in S^{i+1}(K) = S^{i+1}_u(K)$ by the assumption that $B \in S_2$. Therefore, we have $Ant(B) \cup \{B\} \subseteq S^{i+1}_u(K)$. Moreover, we know by definition that $d(B) = j + 1 \leq i + 1$ and since it is clear that

$$B \to B \in Imp(Ant(B) \cup \{B\}, Con(B)),$$

we obtain that $Con(B) \in S^{i+1}_{u+1}(K)|_0 \subseteq S^{i+1}(K)|_0$, which contradicts the assumption that $B \in S_2$.

Case 3. $B \in S_3$. Then, for some $1 \leq m \leq n$, $B = A_m$. Let w_m be the root of $\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i)$. We will show that

$$\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i), w_m \Vdash Ant(A_m), \tag{33}$$

whereas

$$\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i), w_m \nVdash Con(A_m), \tag{34}$$

from which it will follow,by construction of $\mathcal{M}(K)$, that

$$(\mathcal{M}(K), w_m \Vdash Ant(A_m)) \wedge (\mathcal{M}(K), w_m \nVdash Con(A_m))$$

and, finally, that

$$\mathcal{M}(K), w \nVdash A_m.$$

Since $d(A_m) = d(B) = j+1 \leq i+1$, it follows that $d(Ant(A_m)) = j \leq i$. Therefore

$$Ant(A_m) \subseteq (Ant(A_m) \cup S^{i+1}(K)|_i)|_i = S^i_0(Ant(A_m) \cup S^{i+1}(K)|_i),$$

and we have (33) by the choice of $\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i)$.

To show (34) by reductio, we assume that $\mathcal{M}(Ant(A_m) \cup S^{i+1}(K)|_i), w_m \Vdash Con(A_m)$. It follows that $Con(A_m) \in S^i(Ant(A_m) \cup S^{i+1}(K)|_i)$. Now, again using the finitude of $S^{i+1}(K)$, choose a natural u for which $S^{i+1}(K) = S^{i+1}_u(K)$. Then it follows that

$$Con(A_m) \in S^i(Ant(A_m) \cup S^{i+1}_u(K)|_i),$$

which means, by definition, that $A_m \in S^{i+1}_{u+1}(K) \subseteq S^{i+1}(K)$, a contradiction with the assumption that $A_m \in S_3$. □

Corollary 2. *For every $A \in L_{\to}$:*

$$\models A \Leftrightarrow Con(A) \in S^{d(Ant(A))}(Ant(A)).$$

Proof. ($\Leftarrow$). By Lemma 3, if $Con(A) \in S^{d(Ant(A))}(Ant(A)) \subseteq S(Ant(A))$, then $Ant(A) \models Con(A)$. Therefore, $\models A$ by deduction theorem.

($\Rightarrow$). Assume, for reductio, that $\models A$, but $Con(A) \notin S^{d(Ant(A))}(Ant(A))$. If $Con(A) \notin Sub(Ant(A))$ then there exists a model $\mathcal{M}$ consisting of a single world w in which $Con(A)$ fails but $Con(B)$ is satisfied for every $B \in Ant(A)$. It is clear that A fails in this model.

On the other hand, if $Con(A) \in Sub(Ant(A))$ then consider $\mathcal{M}(Ant(A))$. By Lemma 4, if w is the root of $\mathcal{M}(Ant(A))$, and $B \in Sub(Ant(A))$, then

$$\mathcal{M}(Ant(A)), w \Vdash B \Leftrightarrow B \in S^{d(Ant(A))}(Ant(A)).$$

This means, of course, that both

$$\mathcal{M}(Ant(A)), w \Vdash Ant(A),$$

and

$$\mathcal{M}(Ant(A)), w \nVdash Con(A).$$

Therefore, we get that $\mathcal{M}(Ant(A)), w \nVdash A$, which contradicts the assumption that $\models A$. □

Importantly, Lemmas 3 and 4 allow for the following refinement of Corollary 1:

Corollary 3. *For arbitrary finite $K \subseteq L_{\to}$:*

$$S(K) = S^{d(K)}(K).$$

Proof. The inclusion $S(K) \supseteq S^{d(K)}(K)$ holds by definition. For the inverse inclusion, assume that $A \in S(K) \smallsetminus S^{d(K)}(K)$. Then by Lemma 3, A intuitionistically follows from K. On the other hand, $K \subseteq S_0^{d(K)}(K) \subseteq S^{d(K)}(K)$, whereas $A \notin S^{d(K)}(K)$. Thus, by Lemma 4, there is a model $\mathcal{M}(K)$ such that for its root w we have

$$\mathcal{M}(K), w \nVdash \bigwedge K \to A,$$

a contradiction. □

Lemma 5. *For arbitrary finite $K \subseteq L_{\to}$, if $C \to p \in K$, $d(C \to p) = d(K) > 0$, and $C \notin S(K \smallsetminus \{C \to p\})$, then for arbitrary $A \in S(K)$, either there is a set $\Delta \subseteq Sub(K)$ such that*

$$A \in Imp(\Delta \cup \{C\}, p),$$

or

$$K \smallsetminus \{C \to p\} \models A.$$

Proof. Assume the conditions of the Lemma for some K and choose an $A \in S(K) \subseteq Sub(K)$. It follows by Corollary 3, that $A \in S^{d(K)}(K)$. To establish the Lemma, it would suffice to show that for an arbitrary natural j, if $A \in S_j^{d(K)}(K)$ then either there is a set $\Delta \subseteq Sub(K)$ such that

$$A \in Imp(\Delta \cup \{C\}, p),$$

or

$$K \smallsetminus \{C \to p\} \models A.$$

We show this by induction on j.

Basis. Let $j = 0$. Then

$$A \in S_0^{d(K)}(K) = K.$$

Therefore, either $A = C \to p$, or $A \in K \setminus \{C \to p\}$ whence

$$K \setminus \{C \to p\} \models A.$$

Induction hypothesis. Assume that for $j \leq k$ it is true that if $A \in S_j^{d(K)}(K)$ then either for some $\Delta \subseteq Sub(K)$ we have

$$A \in Imp(\Delta \cup \{C\}, p),$$

or

$$K \setminus \{C \to p\} \models A.$$

Induction step. Let $j = k + 1$ and let $A \in S_{k+1}^{d(K)}(K)$. Then three cases are possible:
Case 1. $A \in S_k^{d(K)}(K)$. Then we are done by IH.
Case 2. $A = D \to B$ and

$$Con(B) \in S^{d(K)-1}(S_k^{d(K)}(K)|_{d(K)-1} \cup Ant(B)).$$

Then, by Lemma 3, we have

$$S_k^{d(K)}(K)|_{d(K)-1} \cup Ant(B) \models Con(B),$$

whence, by deduction theorem,

$$S_k^{d(K)}(K)|_{d(K)-1} \models B.$$

We know by IH that for every formula $E \in S_k^{d(K)}(K)|_{d(K)-1}$, either for some $\Delta \subseteq Sub(K)$ we have

$$E \in Imp(\Delta \cup \{C\}, p),$$

or

$$K \setminus \{C \to p\} \models E.$$

But if $E \in Imp(\Delta \cup \{C\}, p)$, then $d(E) = d(C \to p) = d(K)$, therefore we would have $E \notin S_k^{d(K)}(K)|_{d(K)-1}$, which contradicts the choice of E. Hence

$$K \setminus \{C \to p\} \models S_k^{d(K)}(K)|_{d(K)-1},$$

therefore,

$$K \setminus \{C \to p\} \models B,$$

and finally:

$$K \setminus \{C \to p\} \models D \to B = A.$$

Case 3. $A = D \to B$ and there exists $\Gamma \subseteq S_k^{d(K)}$ such that for some $E \in Sub(K)$

$$E \to E \in Imp(\Gamma \cup Ant(B), Con(B))).$$

Now it is clear that $\Gamma \cup Ant(B) = Ant(E) \cup \{E\}$ and $Con(B) = Con(E)$. Therefore, $d(\Gamma \cup Ant(B)) = d(E)$, and since $B \in Sub(K)$ we get that $\Gamma \cup Ant(B) \subseteq Sub(K)$, and so

$$d(E) = d(\Gamma \cup Ant(B)) \leq d(Sub(K)) = d(K).$$

Also, since $d(C \to p) = d(K)$, then for any $\Delta \subseteq Sub(K)$

$$d(K) \leq d(Imp(\Delta \cup \{C\}, p)).$$

All in all, this gives us that for every $\Delta \subseteq Sub(K)$ we have that

$$d(E) \leq d(Imp(\Delta \cup \{C\}, p)).$$

Note, further, that if $E \neq F \in \Gamma \cup Ant(B)$, then we must have $F \in Ant(E)$, and therefore $d(F) < d(E)$ strictly. This means that there is at most one formula $F \in \Gamma \cup Ant(B)$ for which there is a set $\Delta \subseteq Sub(K)$ such that $F \in Imp(\Delta \cup \{C\}, p)$ and, if there is such an F, then $F = E$. So we assume, first, that $E \in Imp(\Delta \cup \{C\}, p)$ for some $\Delta \subseteq Sub(K)$.

Now, in this case $C \in Ant(E) \subseteq \Gamma \cup Ant(B)$. If we have $C \in \Gamma$, then by IH either for some $\Delta \subseteq Sub(K)$ we have

$$C \in Imp(\Delta \cup \{C\}, p),$$

or

$$K \setminus \{C \to p\} \models C.$$

If the latter were true, then by Corollary 2 we would have $C \in S(K \setminus \{C \to p\})$, which contradicts the assumption of the Lemma. If the former were true, then we would have $d(C) > d(C \to p)$ which is an obvious contradiction.

Therefore, we cannot have $C \in \Gamma \subseteq S(K)$, and so $C \in Ant(B)$. This means that

$$((Ant(B) \setminus \{C\}) \cup \{D\}) \cup \{C\} = Ant(D \to B),$$

Furthermore,

$$Con(D \to B) = Con(B) = Con(E) = p.$$

Therefore, for $\Delta = (Ant(B) \setminus \{C\}) \cup \{D\}$ we have

$$D \to B \in Imp(\Delta \cup \{C\}, p),$$

and we are done.

On the other hand, if for every $F \in \Gamma \cup Ant(B)$ there is no set $\Delta \subseteq Sub(K)$ such that $F \in d(Imp(\Delta \cup \{C\}, p))$, then, by IH,

$$K \smallsetminus \{C \to p\} \models \Gamma,$$

and we reason as in Case 2. More precisely, by Lemma 3, we have

$$\Gamma \cup Ant(B) \models Con(B),$$

whence, by deduction theorem, $\Gamma \models B$. Therefore,

$$K \smallsetminus \{C \to p\} \models B,$$

and finally:

$$K \smallsetminus \{C \to p\} \models D \to B.$$

□

Corollary 4. *Let $A \in L_{\to}$, $C \to p \in Ant(A)$, and $d(C \to p) = d(Ant(A)) > 0$. If $\models A$, then one of the following is true:*

$$(1) \quad \models Imp(Ant(A) \smallsetminus \{C \to p\}, Con(A));$$

or

$$(2) \quad C \in S(Ant(A) \smallsetminus \{C \to p\}).$$

Proof. Assume that $C \notin S(Ant(A) \smallsetminus \{C \to p\})$. By $\models A$ and Corollary 2, we know that

$$Con(A) \in S(Ant(A))|_0,$$

and we can apply Lemma 5. Now of course there is no $\Delta \subseteq Sub(K)$ such that we have

$$Con(A) \in Imp(\Delta \cup \{C\}, p),$$

because in this case we would have

$$0 = d(Con(A)) = d(Imp(\Delta \cup \{C\}, p)) = d(C \to p) > 0.$$

Therefore, we must have

$$Ant(A) \smallsetminus \{C \to p\} \models Con(A),$$

whence $\models Imp(Ant(A) \smallsetminus \{C \to p\}, Con(A))$ follows by deduction theorem. □

We use the notation $[A_1 \to \ldots \to A_n]$ for the chain of implications where all parentheses are grouped to the left. For instance, $[A_1 \to A_2 \to A_3]$ stands for $(A_1 \to A_2) \to A_3$. In what follows, we will need to consider a certain group of intuitionistic models. First, consider Kripke frame $\mathcal{F}$ such that:

$$\mathcal{F} = \langle \{w, u, v\}, \{\langle w, w\rangle, \langle v, v\rangle, \langle u, u\rangle, \langle w, v\rangle, \langle w, u\rangle\}\rangle.$$

Then the models that we need to consider below, will look like this:

$$\mathcal{N} = \langle \mathcal{F}, V\rangle;$$
$$\mathcal{N}_n = \langle \mathcal{F}, V_n\rangle$$

where we assume that for every $r \in Var$:

$$V(r) = \begin{cases} \{v\}, \text{ if } r = p_1; \\ \{u, v\}, \text{ if } r = q; \\ \varnothing \text{ otherwise.} \end{cases}$$

$$V_n(r) = \begin{cases} \{v\}, \text{ if } r = p_1; \\ \{u, v\}, \text{ if } r = p_{n+1}; \\ \varnothing \text{ otherwise.} \end{cases}$$

Lemma 6. *Let s be a world in $\mathcal{N}$. Then for every natural n:*

$$\mathcal{N}, s \Vdash [p_1 \to \ldots \to p_n] \Leftrightarrow \begin{cases} n \textit{ is even and } s = u; \\ n \textit{ is odd and } s = v. \end{cases}$$

Proof. We proceed by induction on n.

Basis. One can easily check that the following condition hold:

$$\mathcal{N}, s \Vdash p_1 \Leftrightarrow s = v.$$

Induction hypothesis. Assume that for $n \leq m$ and for an arbitrary world s in $\mathcal{N}$ it is true that:

$$\mathcal{N}, s \Vdash [p_1 \to \ldots \to p_n] \Leftrightarrow \begin{cases} n \text{ is even and } s = u; \\ n \text{ is odd and } s = v. \end{cases}$$

Induction step. Let $n = m + 1$. We can choose a k for which either $m + 1 = 2k + 1$, or $m + 1 = 2k$.

Assume that $m = 2k$; the other case is similar. Then, by IH we have

$$\mathcal{N}, u \Vdash [p_1 \to \ldots \to p_m], \tag{35}$$

and

$$\mathcal{N}, v \nVdash [p_1 \to \ldots \to p_m]. \tag{36}$$

We also have, by definition of $\mathcal{N}$, that

$$\mathcal{N}, u \nVdash p_{m+1}. \tag{37}$$

Now we can infer the following:

$$\mathcal{N}, v \Vdash [p_1 \to \ldots \to p_m] \to p_{m+1}; \qquad \text{(from (36))} \tag{38}$$

$$\mathcal{N}, u \nVdash [p_1 \to \ldots \to p_m] \to p_{m+1}; \qquad \text{(from (35) and (37))} \tag{39}$$

$$\mathcal{N}, w \nVdash [p_1 \to \ldots \to p_m] \to p_{m+1}. \qquad \text{(from (39))} \tag{40}$$

This completes the proof. □

Lemma 7. *For $n \geq 2$, both of the following hold:*

$$\mathcal{N}_n, s \Vdash [p_1 \to \ldots \to p_{n-1}] \Leftrightarrow (s = v \wedge n \textit{ is even}) \vee (s = u \wedge n \textit{ is odd}),$$

and

$$\mathcal{N}_n, w \Vdash [p_1 \to \ldots \to p_{n+1}].$$

Proof. The first part of the Lemma follows from Lemma 6, given that for every natural n, $n - 1$ is even iff n is odd, and $n - 1$ is odd iff n is even, and given the fact that for every n the model $\mathcal{N}_n$ is only different from $\mathcal{N}$ in its valuations for q and p_{n+1} which do not occur in $[p_1 \to \ldots \to p_{n-1}]$.

For the same reasons, it also follows from Lemma 6 that

$$\mathcal{N}_n, s \Vdash [p_1 \to \ldots \to p_n] \Leftrightarrow (s = u \wedge n \text{ is even}) \vee (s = v \wedge n \text{ is odd}). \tag{41}$$

Now for the given n we can always choose a natural k for which either $n = 2k$, or $n = 2k + 1$.

Assume that $n = 2k$; the other case is similar. It follows from (41) that we have:

$$\mathcal{N}_n, w \nVdash [p_1 \to \ldots \to p_n]. \tag{42}$$

We also know that the following equation holds by the definition of $\mathcal{N}_n$:

$$\mathcal{N}_n, s \Vdash p_{n+1} \Leftrightarrow s \in \{u, v\}. \tag{43}$$

Our reasoning is then straightforward:

$$\mathcal{N}_n, u \Vdash [p_1 \to \ldots \to p_{n+1}] \qquad \text{(from (43))} \tag{44}$$

$$\mathcal{N}_n, v \Vdash [p_1 \to \ldots \to p_{n+1}] \qquad \text{(from (43))} \tag{45}$$

$$\mathcal{N}_n, w \Vdash [p_1 \to \ldots \to p_{n+1}] \qquad \text{(from (42), (44), and (45))} \tag{46}$$

□

We need two final pieces of notation: we call A an *n-elimination formula* iff $A = \forall \bar{r} B$, where $B \in L_{(\wedge,\to)}$, $d(B) = n$ and $\bar{r}$ is the list of all variables of B except for possibly $p_1, \ldots, p_n, q$.

Further, we call A an *n-introduction formula* iff $A = B_1 \vee \ldots \vee B_m$, where for every $1 \leq i \leq m$ $B_i \in L_{(\wedge,\to)}$, $d(\{B_1, \ldots, B_m\}) = n$ and all the variables of A are among $p_1, \ldots, p_n$.

We are now ready to formulate and prove our main results:

Theorem 1. *For every natural n, $[p_1 \to \ldots \to p_n] \vee q$ is not intuitionistically equivalent to any n-elimination formula.*

Proof. Assume that an n-elimination formula $A = \forall \bar{r} B$ intuitionistically follows from $[p_1 \to \ldots \to p_n] \vee q$. We will show that

$$\not\models \forall \bar{r} B \to ([p_1 \to \ldots \to p_n] \vee q)$$

First, we can represent B as a conjunction $B_1 \wedge \ldots \wedge B_m$ where $\{B_1, \ldots, B_m\} \subseteq L_\to$ and for every $1 \leq i \leq m$, $d(B_i) \leq n$. We may safely assume that all of $B_1, \ldots, B_m$ are not intuitionistically valid. Indeed, if all of $B_1, \ldots, B_m$ are intuitionistically valid, then $\models A$ and, of course, $[p_1 \to \ldots \to p_n] \vee q$ will not follow from A, so we are done. If only some of $B_1, \ldots, B_m$ are intuitionistically valid, we can simply omit all the valid formulas from this set.

It follows from our assumption that

$$\models ([p_1 \to \ldots \to p_n] \vee q) \to \forall \bar{r}(B_1 \wedge \ldots \wedge B_m),$$

and, further, that

$$\models ([p_1 \to \ldots \to p_n] \vee q) \to \forall \bar{r} B_i,$$

for every $1 \leq i \leq m$.

This, in turn, means that

$$\models [p_1 \to \ldots \to p_n] \to B_i, \tag{47}$$

and

$$\models q \to B_i, \tag{48}$$

again, for every $1 \leq i \leq m$.

Note that since $d(B_i) \leq n$, $d(Ant(B_i)) \leq n - 1$. Therefore,

$$d(Ant([p_1 \to \ldots \to p_n] \to B_i) = d(Ant(B_i) \cup \{[p_1 \to \ldots \to p_n]\}) = n - 1.$$

Since we know that for every $1 \leq i \leq m$, B_i is not intuitionistically valid, it follows from (47) by Corollary 4 that

$$[p_1 \to \ldots \to p_{n-1}] \in S(Ant([p_1 \to \ldots \to p_n] \to B_i) \setminus \{[p_1 \to \ldots \to p_n]\}) = S(Ant(B_i)). \tag{49}$$

Furthermore, by Lemma 6,

$$\mathcal{N}, w \not\Vdash [p_1 \to \ldots \to p_n] \vee q.$$

We fix an arbitrary valuation for the variables A which are distinct from $p_1, \ldots, p_n, q$ and show that under this valuation

$$\mathcal{N}, w \Vdash B_i.$$

Indeed, we know by (48) that both $\mathcal{N}, u \Vdash B_i$ and $\mathcal{N}, v \Vdash B_i$, since q is forced in $\mathcal{N}$ by both of these worlds.

Moreover, it follows from (49) by Lemma 3 that

$$Ant(B_i) \models [p_1 \to \ldots \to p_{n-1}]. \tag{50}$$

We also know, by Lemma 6, that

$$\mathcal{N}, w \not\Vdash [p_1 \to \ldots \to p_{n-1}]. \tag{51}$$

Hence it follows from (50) and (51) that

$$\mathcal{N}, w \not\Vdash Ant(B_i). \tag{52}$$

Given that we have already established that B_i is forced in both successors of w in $\mathcal{N}$, (52) yields that

$$\mathcal{N}, w \Vdash B_i,$$

which completes the proof. □

Theorem 2. *For every $n \geq 1$, $[p_1 \to \ldots \to p_{n+2}]$ is not intuitionistically equivalent to any n-introduction formula.*

Proof. Assume that $[p_1 \to \ldots \to p_{n+2}]$ intuitionistically follows from an n-elimination formula $A = B_1 \vee \ldots \vee B_m$. We will show that

$$\not\models [p_1 \to \ldots \to p_{n+2}] \to (B_1 \vee \ldots \vee B_m)$$

First, by Lemma 1, we can assume that for arbitrary $1 \leq i \leq m$ B_i is actually a conjunction of formulas in $L_{\to}$. Thus for every $1 \leq i \leq m$ we will assume that

$$B_i = C^i_1 \wedge \ldots \wedge C^i_{i_n}.$$

Again, we can assume that every such conjunction is not intuitionistically valid. Indeed, if some of B_is are intuitionistically valid, then A is intuitionistically valid, therefore a non-valid formula like $[p_1 \to \ldots \to p_{n+2}]$ cannot follow from A, a contradiction.

Since $[p_1 \to \ldots \to p_{n+2}]$ follows from A, this means that for every $1 \leq i \leq m$ we have

$$\models B_i \to ([p_1 \to \ldots \to p_{n+1}] \to p_{n+2}).$$

This means that $p_{n+2} \in S(\{C^i_1, \ldots, C^i_{i_n}, [p_1 \to \ldots \to p_{n+1}]\})$.

Now we must have either $[p_1 \to \ldots \to p_n] \in S(\{C^i_1, \ldots, C^i_{i_n}\})$ or $[p_1 \to \ldots \to p_n] \notin S(\{C^i_1, \ldots, C^i_{i_n}\})$. In the latter case, since $d([p_1 \to \ldots \to p_{n+1}]) = n \geq d(\{C^i_1, \ldots, C^i_{i_n}\})$, Corollary 4 applies, and we get that

$$p_{n+2} \in S(\{C^i_1, \ldots, C^i_{i_n}\}).$$

So, in any case either $[p_1 \to \ldots \to p_n] \in S(\{C^i_1, \ldots, C^i_{i_n}\})$, or $p_{n+2} \in S(\{C^i_1, \ldots, C^i_{i_n}\})$. It follows then, by Lemma 3 and the fact that $B_i = C^i_1 \wedge \ldots \wedge C^i_{i_n}$, that

$$B_i \models [p_1 \to \ldots \to p_n] \vee p_{n+2}. \tag{53}$$

Now we know, that since $n + 1 \geq 2$, we have, by Lemma 7, that

$$\mathcal{N}_{n+1}, w \not\Vdash [p_1 \to \ldots \to p_n] \vee p_{n+1}. \tag{54}$$

It is immediate from (53) and (54) that for every $1 \leq i \leq m$ we have

$$\mathcal{N}_{n+1}, w \not\Vdash B_i. \tag{55}$$

Given that we also have, again by Lemma 7, that

$$\mathcal{N}_{n+1}, w \Vdash [p_1 \to \ldots \to p_{n+2}], \tag{56}$$

this completes the proof. □

5 Concluding discussion

Our central result is that there are connectives with level-ℓ introduction rules that do not have harmonious elimination rules of level ℓ or below, and, conversely, connectives with level-ℓ elimination rules that do not have harmonious introduction rules of level ℓ or below. This result could be established for any ℓ greater or equal to one. In a sense it reflects the idea that when passing from introductions to eliminations or from eliminations to introductions in a uniform way we transform premisses into assumptions. When generating $\vee$ elimination

$$\frac{A_1 \vee A_2 \quad \begin{matrix}[A_1] \\ C\end{matrix} \quad \begin{matrix}[A_2] \\ C\end{matrix}}{C}$$

from $\vee$ introduction

$$\frac{A_1}{A_1 \vee A_2} \qquad \frac{A_2}{A_1 \vee A_2}$$

we are turning the premisses A_1 and A_2 of the introduction rules into assumptions in the elimination rule. When we generalise this to a uniform schema for elimination rules

$$\frac{c(A_1, \ldots, A_n) \quad \begin{matrix}[\Delta_1] \\ C\end{matrix} \quad \ldots \quad \begin{matrix}[\Delta_m] \\ C\end{matrix}}{C}$$

generated from introduction rules of the form

$$\frac{\Delta_1}{c(A_1, \ldots, A_n)} \quad \ldots \quad \frac{\Delta_m}{c(A_1, \ldots, A_n)}$$

we again turn the systems Δ_i of rules into assumptions, a procedure, which raises the level by one. Conversely, when generating $\rightarrow$ introduction

$$\frac{\begin{matrix}[A_1] \\ A_2\end{matrix}}{A_1 \rightarrow A_2}$$

from $\rightarrow$ elimination

$$\frac{A_1 \rightarrow A_2 \quad A_1}{A_2}$$

we are turning the minor premiss A_1 of the elimination rule into an assumption of the introduction rule (and its conclusion A_2 into a premiss of the introduction rule). When we generalise this to a uniform schema for introduction rules

$$\frac{\begin{matrix}[\Delta_1] \\ B_1\end{matrix} \quad \ldots \quad \begin{matrix}[\Delta_m] \\ B_m\end{matrix}}{c(A_1, \ldots, A_n)} \quad ,$$

generated from elimination rules of the form

$$\frac{c(A_1, \ldots, A_n) \quad \Delta_1}{B_1} \quad \ldots \quad \frac{c(A_1, \ldots, A_n) \quad \Delta_m}{B_m}$$

we again turn the systems Δ_i of rules, which in the elimination rules occur immediately above the line, into assumptions of the introduction rule. In this way, by going up one level, we can always form harmonious eliminations to given introductions and harmonious introductions to given eliminations. In Schroeder-Heister (2014a) they were called the *canonical elimination rule* (for a given set of introduction rules) and the *canonical introduction rule* (for a given set of elimination rules), since there is only a single such harmonious rule. As the canonical introduction and elimination rule is of higher level than the elimination and introduction rules, respectively, to which they correspond, every connective characterised by a canonical introduction or elimination rule has harmonious elimination or introduction rules, respectively, of lower level. In subsequent work one might ask how to characterise connectives with harmonious introduction and elimination rules, which are of equal (maximum) level, i.e., whose introduction and inference rules are *balanced* in this way. The standard example would be conjunction, but the question is whether there are nontrivial other connectives of this kind.

Whereas our finding that a rise in level cannot always been avoided is a negative result, we should mention the positive aspect of our investigation. By putting formulas of intuitionistic propositional logic in parallel with rules, we could show that to any conjunction-implication formula of degree d there corresponds an introduction rule of level $d+1$ (i.e., with premisses of level d), and to every disjunction of such formulas a set of introduction rules. This means that any connective which is equivalent to a disjunction of conjunction-implication formulas can be given appropriate introduction rules (and, therefore, also a corresponding canonical elimination rule). Likewise, any connective which is equivalent to an arbitrary conjunction-implication formula of degree d can be given appropriate elimination rules of level d (and, therefore, also a corresponding canonical introduction rule). This shows the outstanding role of conjunction-implication formulas for the characterisation of connectives, as such formulas can code what is expressed in terms of rules. In further work this might be extended to formulas also containing universally quantified formulas as proper subformulas, which correspond to quantified higher-level rules (see Schroeder-Heister, 2014a).

References

Dummett, M. (1991). *The Logical Basis of Metaphysics.* London: Duckworth.

Dyckhoff, R. (2009). *Generalised elimination rules and harmony.* (Manuscript, University of St. Andrews, `http://rd.host.cs.st-andrews.ac.uk/talks/2009/GE.pdf`)

Dyckhoff, R. (2014). Some remarks on proof-theoretic semantics. In T. Piecha & P. Schroeder-Heister (Eds.), *Advances in Proof-Theoretic Semantics.* Berlin: Springer.

Francez, N., & Dyckhoff, R. (2012). A note on harmony. *Journal of Philosophical Logic*, *41*, 613–628.

Olkhovikov, G. K., & Schroeder-Heister, P. (2014). On flattening elimination rules. *Review of Symbolic Logic*, *7*, 60–72.

Prawitz, D. (1971). Ideas and results in proof theory. In J. E. Fenstad (Ed.), *Proceedings of the Second Scandinavian Logic Symposium (Oslo 1970)* (pp. 235–308). Amsterdam: North-Holland.

Prawitz, D. (1979). Proofs and the meaning and completeness of the logical constants. In J. Hintikka, I. Niiniluoto, & E. Saarinen (Eds.), *Essays on Mathematical and Philosophical Logic: Proceedings of the Fourth Scandinavian Logic Symposium and the First Soviet-Finnish Logic Conference, Jyväskylä, Finland, June 29 – July 6, 1976* (pp. 25–40 [revised German translation 'Beweise und die Bedeutung und Vollständigkeit der logischen Konstanten, *Conceptus*, 16, 1982, 31–44]). Dordrecht: Kluwer.

Prawitz, D. (2007). Pragmatist and verificationist theories of meaning. In R. E. Auxier & L. E. Hahn (Eds.), *The Philosophy of Michael Dummett* (pp. 455–481). Chicago: Open Court.

Read, S. (2010). General-elimination harmony and the meaning of the logical constants. *Journal of Philosophical Logic*, *39*, 557–576.

Read, S. (2014). General-elimination harmony and higher-level rules. In H. Wansing (Ed.), *Dag Prawitz on Proofs and Meaning.* Heidelberg: Springer.

Schroeder-Heister, P. (1984). A natural extension of natural deduction. *Journal of Symbolic Logic*, *49*, 1284–1300.

Schroeder-Heister, P. (1987). *Structural Frameworks with Higher-Level Rules*. Universität Konstanz: Habil. thesis [see author's homepage].

Schroeder-Heister, P. (2012). Proof-theoretic semantics. In E. Zalta (Ed.), *Stanford Encyclopedia of Philosophy.* Stanford: `plato.stanford.edu`.

Schroeder-Heister, P. (2014a). The calculus of higher-level rules, propositional quantifiers, and the foundational approach to proof-theoretic harmony. *Studia Logica (Special issue, ed. Andrzej Indrzejczak, commemorating the 80th anniversary of Gentzen's and Jaśkowski's groundbreaking works on assumption based calculi).*

Schroeder-Heister, P. (2014b). Generalized elimination inferences, higher-level rules, and the implications-as-rules interpretation of the sequent calculus. In L. C. Pereira, E. H. Haeusler, & V. de Paiva (Eds.), *Advances in Natural Deduction: A Celebration of Dag Prawitz's Work* (pp. 1–29). Heidelberg: Springer.

Schroeder-Heister, P. (2014c). Harmony in proof-theoretic semantics: A reductive analysis. In H. Wansing (Ed.), *Dag Prawitz on Proofs and Meaning.* Heidelberg: Springer.

von Kutschera, F. (1968). Die Vollständigkeit des Operatorensystems $\{\neg, \wedge, \vee, \supset\}$ für die intuitionistische Aussagenlogik im Rahmen der Gentzensemantik. *Archiv für mathematische Logik und Grundlagenforschung*, *11*, 3–16.

Wansing, H. (2000). The idea of a proof-theoretic semantics. *Studia Logica*, *64*, 3–20.

Zucker, J., & Tragesser, R. (1978). The adequacy problem for inferential logic. *Journal of Philosophical Logic*, *7*, 501–516.