DAG Compressions

### L. Gordeev

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# §1. Term compression -1-

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• Dag-complexity (or graph-complexity) of a given algebraic term t is the minimal number of vertices in a rooted DAG (not necessarily a tree!) that represents t. This definition can be constructively specified, as follows.

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- If \$\varsigma\$, \$\vartheta\$ any terms and \$f\$ any function symbol, then
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## Definition

Consider arbitrary algebraic language  $\mathcal{L}$  with individual variables, constants and (w.l.o.g.) binary function symbols. The *terms*:

- Individual variables and constants are terms of the depth 0, also called *atoms*.
- If \$\vec{s}\$, \$t\$ are any terms and \$f\$ any function symbol, then \$f\$ (\$\vec{s}\$,\$t\$) is a term of the depth \$1 + max {depth (\$\vec{s}\$), depth (\$t\$)}; in the Łukasiewicz form we write \$f\$\$\$t\$ instead of \$f\$ (\$\vec{s}\$,\$t\$).

# Term compression -2-

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• 
$$\delta(\mathfrak{t}) := \delta\{\mathfrak{t}\}.$$
  
•  $\delta\{\mathfrak{t}_1, \dots, \mathfrak{t}_k\} := \#\{\mathfrak{t}_1, \dots, \mathfrak{t}_k\}, \text{ if } \mathfrak{t}_1, \dots, \mathfrak{t}_k \text{ are atoms.}$ 

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- $\delta \{f(\mathfrak{s},\mathfrak{t}),\mathfrak{t}_1,\cdots,\mathfrak{t}_k\} := 1 + \delta \{\mathfrak{s},\mathfrak{t},\mathfrak{t}_1,\cdots,\mathfrak{t}_k\}, \text{ if the depth of } \delta \{f(\mathfrak{s},\mathfrak{t}),\mathfrak{t}_1,\cdots,\mathfrak{t}_k\}$ 
  - $f(\mathfrak{s},\mathfrak{t})$  is  $\geq$  than maximal depth of  $\mathfrak{t}_i$ ,  $1 \leq i \leq k$ .

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  - $f(\mathfrak{s},\mathfrak{t})$  is  $\geq$  than maximal depth of  $\mathfrak{t}_i$ ,  $1 \leq i \leq k$ .

For any term t of  $\mathcal{L}$  we define the *dag-complexity* of t,  $\delta$  (t).

$$\delta(\mathfrak{t}) := \delta \{\mathfrak{t}\}.$$

- $\ \, {\bf 0} \ \, \delta\left\{\mathfrak{t}_1,\cdots,\mathfrak{t}_k\right\}:=\#\left\{\mathfrak{t}_1,\cdots,\mathfrak{t}_k\right\}, \, \text{if } \mathfrak{t}_1,\,...,\,\mathfrak{t}_k \text{ are atoms}.$
- $\delta \{f(\mathfrak{s},\mathfrak{t}),\mathfrak{t}_1,\cdots,\mathfrak{t}_k\} := 1 + \delta \{\mathfrak{s},\mathfrak{t},\mathfrak{t}_1,\cdots,\mathfrak{t}_k\}, \text{ if the depth of } f(\mathfrak{s},\mathfrak{t}) \text{ is } \geq \text{ than maximal depth of } \mathfrak{t}_i, 1 \leq i \leq k.$

•  $\delta(\mathfrak{t})$  is easily computable e. g. in Maple.

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# Term compression -3-

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Consider term  $\mathfrak{t} = f(g(x, f(y, h(x, y))), h(x, y))$  of the depth 4 in the language with variables x, y and function symbols f, g, h.

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$$\delta(\mathfrak{t}) = \delta\{\mathfrak{t}\} = 1 + \delta\{g(x, f(y, h(x, y))), h(x, y)\} = 2 + \delta\{x, f(y, h(x, y)), h(x, y)\} = 3 + \delta\{x, y, h(x, y), h(x, y)\} = 3 + \delta\{x, y, h(x, y)\} = 4 + \delta\{x, y, x, y\} = 4 + \delta\{x, y\} = 4 + \#\{x, y\} = 4 + 2 = 6$$

Consider term t = f(g(x, f(y, h(x, y))), h(x, y)) of the depth 4 in the language with variables x, y and function symbols f, g, h. We have

$$\delta(\mathfrak{t}) = \delta\{\mathfrak{t}\} = 1 + \delta\{g(x, f(y, h(x, y))), h(x, y)\}$$
  
= 2 + \delta\{x, f(y, h(x, y)), h(x, y)\}  
= 3 + \delta\{x, y, h(x, y), h(x, y)\}  
= 3 + \delta\{x, y, h(x, y)\} = 4 + \delta\{x, y, x, y\}  
= 4 + \delta\{x, y\} = 4 + \#\{x, y\}  
= 4 + 2 = 6

Note that the ordinary Łukasiewicz length of t is 11.

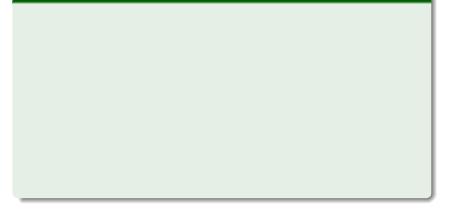
# Term compression -4-

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Define Fibonacci sequence of terms  $\{F(i)\}_{i\geq 0}$  in the language  $\mathcal{L}_{F}$  with two individual constants 0, 1 and one function symbol +;

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$$F(0) := 0, F(1) := 1, F(i+2) := F(i) + F(i+1)$$

Define Fibonacci sequence of terms  $\{F(i)\}_{i\geq 0}$  in the language  $\mathcal{L}_{F}$  with two individual constants 0, 1 and one function symbol +; (use standard infix notation  $\mathfrak{s} + \mathfrak{t}$  instead of Łukasiewicz  $+\mathfrak{st}$ ).

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$$F(0) := 0, F(1) := 1, F(i+2) := F(i) + F(i+1)$$

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- But the corresponding dag-complexity is merely linear in *i* :

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- The ordinary length of F(i) slightly exceeds the  $i^{th}$  Fibonacci number, thus being exponential in i.
- But the corresponding dag-complexity is merely linear in i:  $\delta(F(0)) = \delta(F(1)) = 1$  and  $\delta(F(i)) = i + 1$  for all i > 1.

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Define reductions ▷<sub>0</sub>, ▷<sub>1</sub>, ▷<sub>2</sub> on finite labeled rooted dag's D with reflexive and transitive binary relation R on labels, where D<sub>>y</sub> := the sub-dag of z ≠ y having a path z → y.

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- D ⊳<sub>0</sub> D': D' arises from D by identifying all leaves having the same labels and all vertices x, y such that l(x) = l(y) and (x → y) ∈ D. If not applicable, let D' := D.

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- **2**  $D \triangleright_1 D'$  (contraction),

- Define reductions ▷<sub>0</sub>, ▷<sub>1</sub>, ▷<sub>2</sub> on finite labeled rooted dag's D with reflexive and transitive binary relation R on labels, where D<sub>>y</sub> := the sub-dag of z ≠ y having a path z → y.
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- **2**  $D \triangleright_1 D'$  (contraction), where  $\ell$  is the labeling function:
  - Let x, y be vertices in D closest to the root such that:  $\ell(x) R\ell(y), (x \to y) \notin D, D_{>y}$  is a tree and  $x \in D_{>y}$ .

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  - D' := replace in D whole path  $x \rightsquigarrow y$  by new edge  $x \rightarrow y$ .

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- **3**  $D \triangleright_2 D'$  (conglutination):

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- **(a)**  $D \triangleright_2 D'$  (conglutination):
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  - D' := D plus new edge  $x \to y$  minus  $z \in D_{>y}$ .

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### Definition

•  $\triangleright_i$ -irreducible dag's are called *normal*.

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- 2 Let  $\mathcal{T}$  be set of finite labeled rooted trees, R reflexive and transitive binary relation on labels. For every tree  $T \in \mathcal{T}$  there are chains of dag's  $T = D_0 \triangleright_{i_1} \cdots \triangleright_{i_k} D_k \triangleright_0 D_{k+1}$   $(k \ge 0, i_j = 1, 2)$  with normal  $D_{k+1}$ .

Let #D be standard size of D. Clearlyly every  $\triangleright_i$  is size-reducing:

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- $\triangleright_i$ -irreducible dag's are called *normal*.
- Let T be set of finite labeled rooted trees, R reflexive and transitive binary relation on labels. For every tree T ∈ T there are chains of dag's T = D<sub>0</sub> ▷<sub>i1</sub> ··· ▷<sub>ik</sub> D<sub>k</sub> ▷<sub>0</sub> D<sub>k+1</sub> (k ≥ 0, i<sub>j</sub> = 1, 2) with normal D<sub>k+1</sub>. Clearly #D<sub>k+1</sub> ≤ #T. Call these D<sub>k+1</sub> normal dag-like compressions of T.

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- Set δ(T) := min(#D) for D ranging over normal dag-like compressions of T.

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- Set δ(T) := min(#D) for D ranging over normal dag-like compressions of T. Call δ(T) the dag-complexity of T.

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- Set δ(T) := min(#D) for D ranging over normal dag-like compressions of T. Call δ(T) the dag-complexity of T.
- For any label  $\Gamma$ , let  $\delta(\Gamma) := \min(\delta(T))$  for T ranging over  $T \in T$  with root-label  $\Gamma$ .

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### Definition

- $\triangleright_i$ -irreducible dag's are called *normal*.
- Let T be set of finite labeled rooted trees, R reflexive and transitive binary relation on labels. For every tree T ∈ T there are chains of dag's T = D<sub>0</sub> ▷<sub>i1</sub> ··· ▷<sub>ik</sub> D<sub>k</sub> ▷<sub>0</sub> D<sub>k+1</sub> (k ≥ 0, i<sub>j</sub> = 1, 2) with normal D<sub>k+1</sub>. Clearly #D<sub>k+1</sub> ≤ #T. Call these D<sub>k+1</sub> normal dag-like compressions of T.
- Set δ(T) := min(#D) for D ranging over normal dag-like compressions of T. Call δ(T) the dag-complexity of T.
- So For any label Γ, let δ(Γ) := min(δ(Τ)) for T ranging over T ∈ T with root-label Γ. Call δ(Γ) the dag-complexity of Γ.

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### Problem

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1. How to compute  $\delta(T)$  and/or  $\delta(\Gamma)$  ?

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  - Term algebra  $(\delta(T) \cong \delta(\Gamma)$ , see Chapter 1)

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  - Term algebra ( $\delta(T) \cong \delta(\Gamma)$ , see Chapter 1)
    - Problem 1: Easy (see Chapter 1).

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  - Term algebra ( $\delta(T) \cong \delta(\Gamma)$ , see Chapter 1)
    - Problem 1: Easy (see Chapter 1).
    - **Problem 2**: Roughly  $\#T \ge \delta(T) \ge \log \#T$ .

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- 2. How to estimate  $\delta(T)$  and/or  $\delta(\Gamma)$  ?
  - Term algebra ( $\delta(T) \cong \delta(\Gamma)$ , see Chapter 1)
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  - Generalizations: Both problems are hard.

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- 2. How to estimate  $\delta(T)$  and/or  $\delta(\Gamma)$  ?
  - Term algebra ( $\delta(T) \cong \delta(\Gamma)$ , see Chapter 1)
    - Problem 1: Easy (see Chapter 1).
    - Problem 2: Roughly #T ≥ δ(T) ≥ log #T. In most interesting cases #T exponential in δ(T) (see Chapter 1).
  - **Generalizations**: Both problems are hard. Proof theory provides most interesting applications.

### Proof-theoretic interpretation

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• Let S be given finite collection of axioms and inference rules, as usual in proof theory.

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- Let *T* and *D* contain resp. *tree-like* and *dag-like proofs* (or *deductions*) as *S*-generated trees, resp. dag's, labeled by, say, *sequents* (Γ, Γ', etc.).

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- Let  $\Gamma R\Gamma' :\Leftrightarrow \Gamma' = \theta(\Gamma)$  for  $\theta \in Hom(Seq \rightarrow Seq)$ ,

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### Proof search connections

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  - However D may depend on the choice of ▷<sub>2</sub> involved; thus the sources T can have different normal forms.

### Propositional logic

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Classical propositional logic.

- Classical propositional logic.
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  - **Example**: Very efficient sequent calculus for DNF tautologies, called SEQ $_{\rm TAU}$ .

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• Sequents:  $\Gamma = M_1, \cdots, M_s$  where  $M_i \subset_{\text{fin}} \mathbb{Z}_0 := \mathbb{Z} - \{0\}$  such that  $(\forall a, b \in M_i) (a + b \neq 0)$ 

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• Relation  $R := \{W_1, W_2, S\}^*$  (transitive closure)

# $SEQ_{TAU}$ : Some special cases

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### $SEQ_{TAU}$ : Some special cases

• "Perfect" special case of weakening  $(W_0)$  :

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• "Perfect" special case of Q whose side sequent ( $\Gamma$ ) is empty, i.e. the following rule  $Q_0$  :

$$\begin{array}{ll} \underline{M_1, \cdots, M_r} & \underline{M'_1, \cdots, M'_{r'}} \\ \overline{\{k\} \cup M_1, \cdots, \{k\} \cup M_r, \{-k\} \cup M'_1, \cdots, \{-k\} \cup M'_{r'}} \\ \text{where } \left( \forall 1 \le i \le r, 1 \le j \le r' \right) \left( \pm k \notin M_i, M'_j \right) \end{array}$$

# SEQ<sub>TAU</sub>: Examples

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# $\mathsf{SEQ}_{\mathrm{TAU}}: \ \mathsf{Examples}$

### Example

$$\frac{(S)\frac{\{1\},\{-1\}}{\{2\},\{-2\}} \xrightarrow{(S)} \{4\},\{-4\}}{\{3\},\{-3,4\},\{-3,-4\}}(Q)}{\{1,2\},\{1,-2\},\{-1,3\},\{-1,-3,4\},\{-1,-3,-4\}}(Q)$$

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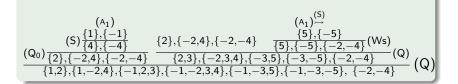
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### $\mathsf{SEQ}_{\mathrm{TAU}}: \ \mathsf{Examples}$

#### Example

$$\frac{(\mathsf{S}) \xrightarrow{\{1\}, \{-1\}} \xrightarrow{(\mathsf{S})} \{4\}, \{-4\}}{\{2\}, \{-2\}} (\mathsf{Q})}{\{1, 2\}, \{1, -2\}, \{-1, 3\}, \{-1, -3, 4\}, \{-1, -3, -4\}} (\mathsf{Q})}$$

### Example



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 Γ is tree-like provable in SEQ<sub>TAU</sub> iff Γ is dag-like provable in SEQ<sub>TAU</sub>.

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# Semantics of SEQ<sub>TAIL</sub>

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Proof.				
Easy.				
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# More on $\mathsf{SEQ}_{\mathrm{TAU}}$

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# More on $\mathsf{SEQ}_{\text{TAU}}$

• Well-known "hard" tautologies have polynomial size dag-like proofs in SEQ<sub>TAU</sub> obtained by basic proof search (see below).

These examples include e.g.:

Doubling names tautologies by Takeuti and Statman.

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#### Theorem

There are  $\Gamma \in TAU$  such that for all tree-like deductions T of  $\Gamma$ , #T is exponential in  $\#\Gamma$ , whereas  $\delta(\Gamma)$  is polynomial in  $\#\Gamma$ .

## Reminder: Clique coloring principle

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#### Theorem

### Clique coloring principle:

No n-element graph G, |G| = n, has a (k - 1)-colored k-element clique  $K \subseteq G$  such that  $2 \le k = |K| \le n$  and there is no edge (in G) between any pair of vertices (in K) having the same color.

# Basic dag-like proof search in $SEQ_{TAU}$

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## Basic dag-like proof search in $\mathsf{SEQ}_{\text{TAU}}$

Consider any given sequent  $\Gamma_0$ . Starting with  $\Gamma_0$  reduce sequents by inverting the rules (W<sub>0</sub>) and (Q) repeatedly, while simultaneously analyzing pairs of new sequents  $\Gamma_i$ ,  $\Gamma_j$  thus obtained which are not axioms and occur in different branches:

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This reduction procedure terminates. Consider the resulting sequent dag D and let  $D \succ_0 D'$ .

If all leaves of D are axioms, then D' is a desired dag-like deduction of  $\Gamma$ . Otherwise  $\Gamma$  is invalid.

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• It is well-known that adding *cut rule* to cutfree proof systems can exponentially accelerate propositional provability (re: propositional *speed-up*). However proof systems with (CUT) or *modus ponens* or similar non-analytic inferences, known as general *Frege systems*, don't admit reasonable poor search.

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- By familiar cut-elimination arguments, any Frege system is reducible to tree-like, and hence also dag-like version of SEQ<sub>TAU</sub> without substitution. Can analogous cut elimination with substitution be done with sub-exponential growth of the resulting dag-like deductions in SEQ<sub>TAU</sub>?

## "Academic" Conjectures C1, C2

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## "Academic" Conjectures C1, C2

## Definition

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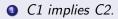
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- C1 implies C2.
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#### Theorem

C1 implies C2.

**2** *C2 implies* NP = coNP.

### Proof.

## Clear.

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#### Lemma

$$TAU_0^{(n)} \in NP.$$

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$$TAU_0^{(n)} \in NP.$$

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Plausible Conjecture C3:

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#### Theorem

C3 implies P < NP.