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PROOF-THEORETIC VALIDITY AND THE COMPLETENESS OF INTUITIONISTIC LOGIC

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1. INTRODUCTION

As has often been claimed, the introduction (I) and elimination (E) rules of intuitionistic natural deduction systems stand in a certain harmony with each other. This can be understood in such a way that once the I rules are given the E rules are uniquely determined and vice versa. The following is an attempt to elaborate this claim. More precisely, we define two notions of validity, one based on I rules (valid+) and one based on E rules (valid-), and show: the E rules generate a maximal valid+ extension of the I rules, and the I rules generate a maximal valid- extension of the E rules. That is to say, the calculus consisting of I and E rules (i.e., intuitionistic logic) is sound and complete with respect to both validity+ and validity-. This does not mean that the syntactical form of E rules is determined by the I rules and vice versa, but that the deductive power which E rules bring about in addition to I rules is exactly what can be justified from I rules and vice versa. Concerning the approach based on I rules this was already claimed by Gentzen who considered I rules to give meanings to the logical signs and the E rules to be consequences thereof (Gentzen, 1935, p. 189).

Unfortunately (and contrary to the author's expectation) the approach based on I rules does not work for the full system of intuitionistic logic, but only for the fragment without v (and ∃ in the quantifier case). The reason for this asymmetry between the two approaches is the indirect character of the E rules for v (and ∃), as will be shown at the end (see section 4.2 below). Thus from the standpoint of the present investigation preference must be given to the approach based on E rules. This, however, is only a technical argument. Whether one should use I or E rules as the basis of proof-theoretic validity must finally rest on genuine semantical or epistemological arguments which go beyond the scope of this paper.

It is a central feature of the systematics of I and E rules of intuitionistic logic, that an I step followed by an E step whose major premiss is just the conclusion of the I step represents a 'detour' and can be omitted: the premisses of the I step already 'contain' the conclusion of the E step in a certain sense. In Prawitz (1965) this fact is called the 'inversion principle' and used as the basis of the normalization procedures developed there. For example, the sequence of inference steps

$$\&E\ \dfrac{\&I\ \dfrac{\begin{matrix}\cdot\\ \cdot\\ \cdot\\ \alpha\end{matrix}\qquad\begin{matrix}+\\ +\\ +\\ \beta\end{matrix}}{\alpha\ \&\ \beta}}{\alpha}$$

can be reduced to

$$\begin{matrix}\cdot\\ \cdot\\ \cdot\\ \alpha,\end{matrix}$$

the sequence

$$
\begin{array}{llll}
 & \cdot & \overline{\alpha}^{(1)} & \overline{\beta}^{(1)} \\
 & \cdot & | & + \\
 & \cdot & | & + \\
 & \alpha & | & + \\
\vee I & \overline{\alpha \vee \beta} & \gamma & \gamma \\
\vee E & \multicolumn{3}{l}{\overline{}\ (1),} \\
 & \gamma & &
\end{array}
$$

where "(1)" marks the discharging of assumptions by the application of $\vee$E, can be reduced to

$$
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\alpha \\
| \\
| \\
| \\
\gamma
\end{array}
$$

(here the derivations of the minor premisses of $\vee$E must be at one's disposal). Using the concept of the eliminability of a rule from a derivation, which means that the derivation which possibly applies that rule can be transformed into a derivation of the same formula from the same or from fewer assumptions without any application of that rule, these reduction steps can be characterized in two ways:

(i) They show that applications of E rules are eliminable from all derivations in which major premisses of such applications are always conclusions of applications of I rules.

(ii) They show that applications of I rules are eliminable from all derivations in which the conclusions of such applications are always major premisses of applications of E rules.

If one considers eliminability in this sense to be a justification of inference rules, in the first case one takes the elimination procedure to be a justification of the E rules with respect to given I rules (which are considered 'canonical', i.e., justified by definition as meaning-determining rules), and in the second case the same procedure to be a justification of the I rules with respect to given 'canonical' E rules.

This idea can easily be generalized to definitions of validity for arbitrary inference rules ρ of the form

$$\frac{\begin{matrix}\Gamma_1 & & \Gamma_n \\ \alpha_1 & \ldots & \alpha_n\end{matrix}}{\alpha} \quad (n \geq 0),$$

where the Γ_i are (possibly empty) lists of assumptions which may be discharged by the application of ρ in the derivation of α_i from Γ_i. Calling all α_i for which Γ_i is empty the principal premisses of ρ (and correspondingly for applications of ρ) one only has to use what is said in (i) and (ii) about E and I rules, respectively, as the definiens. That is to say, we consider arbitrary inference rules ρ as if they were E rules for their principal premises, or as if they were I rules for their conclusion.

(i') ρ is valid+ if applications of ρ are eliminable from all derivations in which the non-atomic principal premisses of such applications are always conclusions of applications of I rules.

(ii') ρ is valid- if applications of ρ are eliminable from all derivations in which the non-atomic conclusions of applications of ρ are always major premisses of applications of E rules.

These definitions have the disadvantage that validity is not transitive in the sense that rules derived by application of valid rules are valid. E.g., both

$$\frac{(\alpha \;\&\; \beta) \;\&\; \gamma}{\alpha \;\&\; \beta} \quad \text{and} \quad \frac{\alpha \;\&\; \beta}{\alpha}$$

are valid+, but not

$$\frac{(\alpha \;\&\; \beta) \;\&\; \gamma}{\alpha},$$

since there is no elimination procedure for the application ⊠ of the latter rule in the derivation

$$(1) \qquad ⊠\ \frac{\dfrac{\overline{\alpha \;\&\; \beta} \qquad \overline{\gamma}}{(\alpha \;\&\; \beta) \;\&\; \gamma}}{\alpha}$$

where $\alpha \;\&\; \beta$ and γ are used as assumptions. Similarly, both

$$\frac{\alpha \qquad \beta}{\alpha \;\&\; \beta} \quad \text{and} \quad \frac{\alpha \;\&\; \beta \qquad \gamma}{(\alpha \;\&\; \beta) \;\&\; \gamma}$$

are valid-, but not

$$\frac{\alpha \qquad \beta \qquad \gamma}{(\alpha \;\&\; \beta) \;\&\; \gamma},$$

since there is no elimination procedure for the application ⊠ of the latter rule in the derivation

$$(2) \quad \boxtimes \quad \dfrac{\dfrac{\overline{\alpha} \quad \overline{\beta} \quad \overline{\gamma}}{(\alpha \,\&\, \beta) \,\&\, \gamma}}{\alpha \,\&\, \beta}$$

where α, β, γ are assumptions. Another example is the inference rule

$$\dfrac{\alpha \qquad \begin{matrix} \alpha \vee \beta \\ \alpha \,\&\, \beta \end{matrix}}{\alpha \,\&\, \beta}$$

(where occurrences of $\alpha \vee \beta$ can be discharged by the application of this rule): it is not valid- in the sense of (ii') because it cannot for example be eliminated from the derivation

$$(3) \quad \boxtimes \quad \dfrac{\dfrac{\overline{\alpha} \qquad \dfrac{\overline{(\alpha \vee \beta) \supset (\alpha \,\&\, \beta)} \quad \overline{\alpha \vee \beta}^{\,(1)}}{\alpha \,\&\, \beta}}{\alpha \,\&\, \beta}^{\,(1)}}{\beta} \, .$$

It can nevertheless be derived by use of the valid- rule

$$\dfrac{\alpha}{\alpha \vee \beta} \, :$$

from $\begin{matrix} + \\ + \\ + \\ \alpha \end{matrix}$ and $\begin{matrix} \alpha \vee \beta \\ \vdots \\ \alpha \,\&\, \beta \end{matrix}$ one obtains $\begin{matrix} + \\ + \\ + \\ \dfrac{\alpha}{\alpha \vee \beta} \\ \vdots \\ \alpha \,\&\, \beta \end{matrix}$.

Thus the soundness and completeness results which can be proved for intuitionistic logic with respect to the notions of validity defined by (i') and (i'') (cf. Schroeder-Heister, 1983a, 1983b) only lead to an equivalence between derivability in intuitionistic logic and the 'transitive closures' of valid rules; this gives much less information than one would like to have (that is reflected in the fact that the proofs of these results are fairly simple).

This difficulty can be overcome by defining the validity of an inference rule ρ by induction on the complexity of ρ (which will be explicated by the rank $|\rho|$ of ρ). We allow that in eliminating an application of ρ from a derivation, rules which are of lower complexity than ρ may be used. This leads to the following preliminary definitions (for precise definitions see section 2.4 below):

(i'') An atomic rule is valid+ if it is derivable without use of basic rules for operators. A non-atomic rule ρ is valid+ if applications of ρ are eliminable from all derivations in which the non-atomic principal premisses of applications of ρ are always conclusions of applications of I rules, and where, besides I rules, valid+ rules of lower complexity than ρ are at one's disposal.

(ii'') An atomic rule is valid- if it is derivable without use of basic rules for operators. A non-atomic rule ρ is valid- if applications of ρ are eliminable from all derivations in which the non-atomic conclusions of applications of ρ are always major premisses of applications of E rules, and where, besides E rules, valid- rules of lower complexity than ρ are at one's disposal.

Using these definitions the applications $\boxtimes$ of the rules considered in the examples above can be eliminated from (1), (2) and (3): We may transform (1) to

$$\cfrac{\cfrac{}{\alpha \ \& \ \beta}}{\alpha}$$

because $\frac{\alpha \ \& \ \beta}{\alpha}$ is a valid+ rule of lower complexity than the rule $\frac{(\alpha \ \& \ \beta) \ \& \ \gamma}{\alpha}$. (2) can be transformed to

$$\cfrac{\cfrac{}{\alpha} \qquad \cfrac{}{\beta}}{\alpha \ \& \ \beta}$$

because $\frac{\alpha \quad \beta}{\alpha \ \& \ \beta}$ is a valid- rule of lower complexity than the rule $\frac{\alpha \quad \beta \quad \gamma}{(\alpha \ \& \ \beta) \ \& \ \gamma}$. Similarly, we may transform (3) to

$$\cfrac{\cfrac{\cfrac{}{(\alpha \vee \beta) \supset (\alpha \ \& \ \beta)} \qquad \cfrac{\cfrac{}{\alpha}}{\alpha \vee \beta}}{\alpha \ \& \ \beta}}{\beta}$$

because $\frac{\alpha}{\alpha \vee \beta}$ is a valid- rule of lower complexity than

$$\frac{\alpha \qquad \begin{matrix}\alpha \vee \beta \\ \alpha \ \& \ \beta\end{matrix}}{\alpha \ \& \ \beta} \, .$$

It can however be shown that the rule

$$\frac{(\alpha \vee \beta) \ \& \ \gamma}{(\alpha \ \& \ \gamma) \vee (\beta \ \& \ \gamma)} \, ,$$

for example, is not valid+ in the sense of (i''), because to eliminate it from certain derivations one would have to use an vE rule

$$\frac{\alpha \vee \beta \qquad \begin{matrix}\alpha \\ (\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)\end{matrix} \qquad \begin{matrix}\beta \\ (\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)\end{matrix}}{(\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)}$$

which is valid+, but not of lower complexity than the rule above (see section 5.2 below). (i'') will turn out to be adequate only for rules which are built up from formulas without v (and ∃ in the quantifier case). Thus when speaking about validity+ this should always be understood as restricted in this way.

The fact that the so-defined notions of validity are in general transitive, follows immediately from the soundness and completeness results we are going to prove with respect to these notions, i.e., from the fact that a rule ρ is derivable in intuitionistic logic iff it is valid+ and iff it is valid-. Due to lack of space we confine ourselves to sentential logic; the quantifier case is an exercise which gives no new fundamental insight. We shall, however, take atomic bases into account, i.e., consider calculi for atomic formulas which are extended with rules for logically compound formulas.

In the following we shall understand the term 'eliminability' as employed above in its uniform reading. That is to say, the elimination procedures which transform certain derivations of the premisses of a rule into certain derivations of its conclusion must be uniform in the sense that they do not depend on the way the premisses have been derived. This idea will be formally captured by introducing assumption rules, i.e., rules functioning as assumptions, which are a natural counterpart of assumption formulas for natural deduction systems.

Section 2 below will present our system of sentential logic over atomic calculi including assumption rules in detail, more precise re-definitions of validity, and some basic lemmata. Sections 3 and 4 give the soundness and completeness proofs for validity- and validity+, respectively, and section 5 contains, beside some remarks on the extension of our approach to quantifier logic, a discussion of why the conception which is based on I rules fails for formulas containing $\vee$ (or $\exists$).

The approach presented here was stimulated by and is closely related to Prawitz' theory of arguments and his definitions of validity (cf. Prawitz, 1971, 1973, 1974, 1984). Because we are concerned mainly with technical matters of proof theory, we always speak of 'derivations' and do not distinguish them from 'arguments'. A comparison of ours and Prawitz' approach cannot be carried out here; we just mention some aspects which must be taken into consideration for such a comparison:

- Prawitz defines the validity of inference rules in terms of the validity of arguments which is the primary notion. We define validity of an inference rule first; a derivation is considered valid if it results from applications of valid inference rules.

- Prawitz' definitions of validity which are based on I rules capture the whole system of intuitionistic logic whereas our notion of validity+ is adequate only for the restricted system without $\vee$ (and $\exists$). Conversely, Prawitz' short discussion of a validity notion based on E rules (1971, p. 289 seq.) is restricted to the system without $\vee$ (and $\exists$) whereas our notion of validity- is adequate for the full system.

- Prawitz leaves the notion of a 'procedure' unrestricted, whereas we will consider only uniform procedures. It seems to be exactly this restriction which makes the fairly simple completeness proofs for our system possible (whereas the soundness proofs are more complicated because they have to apply the whole power of the normalization theorem). For Prawitz' approach there is an obvious way of proving soundness but it is not at all clear how a completeness proof may proceed. Furthermore, our restriction to uniform procedures is the reason why, when dealing with atomic systems, we need not consider arbitrary extensions of such systems in the definitions of validity.

2. BASIC NOTIONS

2.1 Calculi With Assumption Rules - Uniform Procedures

As stated above we deal with sentential logic which is built over atomic calculi for atomic formulas. An atomic calculus A is given by a set of atomic formulas and a set of inference rules (perhaps including axioms) governing these formulas. The calculus of intuitionistic logic $I(A)$ over A is defined as follows: Formulas of $I(A)$ are formulas of A, and furthermore $(\alpha \& \beta)$, $(\alpha \vee \beta)$, $(\alpha \supset \beta)$, $\bot$ for formulas α and β of $I(A)$, where outer brackets can be omitted. Such formulas are also called *formulas over* A. For given A, 'α', 'β', 'γ', 'δ', 'ε' (with and without indices) are syntactical variables for formulas over A. Formulas of $I^{o}(A)$ are formulas over A which do not contain $\vee$. *Basic inference rules* (shortly: basic rules) of $I(A)$ are the atomic inference rules of A and the standard I and E rules of intuitionistic sentential logic, i.e.,

&I $\dfrac{\alpha \quad \beta}{\alpha \& \beta}$ &E $\dfrac{\alpha \& \beta}{\alpha} \quad \dfrac{\alpha \& \beta}{\beta}$

$$\vee I \quad \frac{\alpha}{\alpha \vee \beta} \quad \frac{\beta}{\alpha \vee \beta} \qquad \vee E \quad \frac{\alpha \vee \beta \quad \begin{matrix}\alpha \\ \gamma\end{matrix} \quad \begin{matrix}\beta \\ \gamma\end{matrix}}{\gamma}$$

$$\supset I \quad \frac{\begin{matrix}\alpha \\ \beta\end{matrix}}{\alpha \supset \beta} \qquad \supset E \quad \frac{\alpha \supset \beta \quad \alpha}{\beta}$$

[no $\bot$I rule] $\qquad \bot E \quad \frac{\bot}{\alpha}$

for all α, β, γ. Basic rules of $I^{o}(A)$ are these rules without the $\vee$ rules, where α, β, γ are formulas of $I^{o}(A)$. Note that rules in our sense are not rule schemata so there are, e.g., many $\supset$I rules and not one $\supset$I rule. The formulas standing above a premiss of a rule are those whose occurrences may be discharged by the application of this rule. In the notation of a derivation this discharging must be indicated (e.g., by small numerals in brackets), but not in the notation of a rule, for in the latter case assumptions which are not discharged are never mentioned. As usual, the leftmost formulas of E rules and of applications of E rules are called major premisses.

$I^{+}(A)$ results from $I^{o}(A)$ by omitting the E rules, $I^{-}(A)$ from $I(A)$ by omitting the I rules as basic rules. By I we denote the calculus $I(F)$ over the atomic calculus F where F has denumerably many sentence letters a_1, a_2, ... as formulas and no inference rules. I.e., I is the ordinary calculus of 'formal' intuitionistic sentential logic without an atomic base in the genuine sense. By I^{o}, I^{+} and I^{-} we denote $I^{o}(F)$, $I^{+}(F)$ and $I^{-}(F)$, respectively.

Following Lorenzen (1955), by a *system* of signs we understand a list of signs whose entries are separated by commas and called its members. Note that α is just the system consisting of α only (which is different from the finite set $\{\alpha\}$). If Φ is a system, then $\{\Phi\}$ is the set having as elements the members of Φ. Empty systems are allowed as limiting cases - they have no graphical representation. Syntactical variables for systems of formulas are 'Γ', 'Δ', with and without indices. 'Γ, Δ' denotes the system which is obtained by concatenating the members of Γ and Δ. It will be always obvious from the context, if the comma is used as dividing entries of a system or if it is metalinguistically used in the context of an enumeration.

The general form of an inference rule (shortly: rule) is

$$(4) \quad \frac{\begin{matrix}\Gamma_1 & & \Gamma_n \\ \alpha_1 & \dots & \alpha_n\end{matrix}}{\alpha}$$

where Γ_i's may be empty and n may be 0. Syntactical variables for rules are 'ρ', 'ρ_1', 'ρ_2' , ... Rules over A (or 'of $I(A)$') are built up from formulas over A, rules of $I^o(A)$ from formulas over A without v. As a linear notation for a rule ρ of the form (4) we use

$$<<\Gamma_1> \Rightarrow \alpha_1, \ \dots, \ <\Gamma_n> \Rightarrow \alpha_n> \Rightarrow \alpha$$

where, if Γ_i is empty, $<\Gamma_i> \Rightarrow \alpha_i$ is replaced by α_i and where, if $n = 0$, we write $\Rightarrow\alpha$. By $(\rho)_1$ we denote the system

$$<\Gamma_1> \Rightarrow \alpha_1, \ \dots, \ <\Gamma_n> \Rightarrow \alpha_n$$

and by $(\rho)_2$ the conclusion α. In this linear notation, we write, e.g., $<\alpha, \beta> \Rightarrow \alpha \ \& \ \beta$ for an &I rule and $<\alpha \vee \beta, \ <\alpha> \Rightarrow \gamma, \ <\beta> \Rightarrow \gamma> \Rightarrow \gamma$ for an vE rule. As syntactical variables for systems containing formulas and/or rules of the form $<\Gamma> \Rightarrow \alpha$ we use 'Φ' and 'Ψ' (with and without indices). (Note that Φ and Ψ cannot contain arbitrary rules of the form (4) but only those where all Γ_i are empty.) So a rule ρ of the general form (4) can be represented as $<\Phi> \Rightarrow \alpha$.

The rank of formulas, rules and systems of formulas and rules is denoted by '$|\ |$' and defined as follows:

$|\alpha| = 0$ if α is atomic

$|\bot| = 1$

$|\alpha \,\&\, \beta| = |\alpha \vee \beta| = |\alpha \supset \beta| = \max(|\alpha|, |\beta|) + 1$

$|\Phi| = \max\{|\Psi| / \Psi \text{ is member of } \Phi\}$

$|<\Phi> \Rightarrow \alpha| = \max(|\Phi|, |\alpha|) + 1.$

For example, according to this definition, $\vee$E rules and $\supset$I rules have rank 2 if α, β, γ are all atomic. Note that the rank of a rule is always greater than 0.

Lemma 2.1.1: (i) An I rule with conclusion α has rank $|\alpha| + 1$. The same holds for E rules with α as major premiss, if α is a conjunction or an implication.

(ii) If $|\beta| < |\alpha|$, then $\max(|<\Phi> \Rightarrow \beta| - 1, |\beta| + 1) \leq$
$\leq |<\Phi> \Rightarrow \alpha| - 1$.

Proof (i) trivial.

(ii) $|<\Phi> \Rightarrow \beta| - 1 = \max(|\Phi|, |\beta|) \leq \max(|\Phi|, |\alpha|) =$
$= |<\Phi> \Rightarrow \alpha| - 1$. $|\beta| + 1 \leq |\alpha| \leq \max(|\Phi|, |\alpha|) = |<\Phi> \Rightarrow \alpha| - 1$.

As usual in natural deduction systems, derivations may start with assumptions (or applications of atomic axioms if there are any) and proceed by the application of inference rules where by the application of certain rules assumptions can be discharged, i.e., the dependence on assumptions can be eliminated in the course of a derivation. Unlike usual treatments of natural deduction, however, we permit the application of rules of the form $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ as additional assumptions. These are not assumptions from which one can start, but assumptions which are applied according to the schema

$$\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha \quad \dfrac{\begin{matrix}\vdots & & \vdots \\ \alpha_1 & \ldots & \alpha_n\end{matrix}}{\alpha}$$

and allow one to pass over from derived formulas to another formula. Assumption rules differ from assumption formulas in that, in the framework presented here, there are no rules which allow one to discharge applications of them. We use the term 'assumption' as including assumption formulas and assumption rules. (For a thorough treatment of assumption rules within a framework where they are dischargeable, including the quantifier case, see Schroeder-Heister 1984a, 1984b).

Assumption rules are a natural extension of assumption formulas for the context of natural deduction systems in so far as they can help to explicate the notion of a *uniform* procedure for transforming derivations from assumption formulas; such a notion is important for the definition of the derivability of a rule which allows us to discharge assumptions. Following Lorenzen (1955), in the case of Hilbert-style calculi one can distinguish between the admissibility and the derivability of a rule $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ in the following way: $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ is *admissible* if there is a procedure which transforms assumption-free derivations of $\alpha_1, \ldots, \alpha_n$ into an assumption-free derivation of α; it is *derivable* if there is a uniform procedure of this kind, i.e., a procedure which does not depend on the way the $\alpha_1, \ldots, \alpha_n$ may have been derived. The latter can be expressed by saying that α is derivable *from* the assumptions $\alpha_1, \ldots, \alpha_n$. Occurrences of the assumptions $\alpha_1, \ldots, \alpha_n$ in a derivation of α from $\alpha_1, \ldots, \alpha_n$ can be considered as schematic letters representing assumption-free derivations of $\alpha_1, \ldots, \alpha_n$, respectively: instantiation of them by assumption-free derivations uniformly yields an assumption-free derivation of α.

The same distinction can be drawn on the next level with natural deduction calculi. A rule of the form $\langle\langle\Gamma_1\rangle \Rightarrow \alpha_1, \ldots, \langle\Gamma_n\rangle \Rightarrow \alpha_n\rangle \Rightarrow \alpha$ may be called admissible, if there is a procedure which transforms derivations of α_i from Γ_i (for all i, $1 \leq i \leq n$) into an assumption-free derivation of α, and called derivable if there is a uniform procedure of this kind, i.e., a procedure which does not depend on the way the α_i may have been derived from the Γ_i. The latter can be expressed by saying that α is derivable *from* the assumptions $\langle\Gamma_1\rangle \Rightarrow \alpha_1, \ldots, \langle\Gamma_n\rangle \Rightarrow \alpha_n$. Applications of assumptions $\langle\Gamma_i\rangle \Rightarrow \alpha_i$ in such a derivation can be considered as schematically representing derivations of α_i from Γ_i: replacement of them by derivations of α_i from Γ_i uniformly yields an assumption-free derivation of α.

Consider for example the rule

$$\frac{\begin{array}{cc} \alpha & \beta \\ \beta \,\&\, \gamma & \delta \end{array}}{\alpha \supset \delta}$$

for arbitrary formulas α, β; γ, δ of *I*. Its derivability in *I* can be shown using assumption rules in the following way:

$$
\begin{array}{rcl}
 & \overline{} & (1) \\
 & \alpha & \\
\langle\alpha\rangle \Rightarrow \beta \,\&\, \gamma & \overline{} & \\
 & \beta \,\&\, \gamma & \\
\&E & \overline{} & \\
 & \beta & \\
\langle\beta\rangle \Rightarrow \delta & \overline{} & \\
 & \delta & \\
\supset I & \overline{} & (1). \\
 & \alpha \supset \delta &
\end{array}
$$

This derivation expresses a uniform procedure because the replacement of

$$\langle\alpha\rangle \Rightarrow \beta \,\&\, \gamma \quad \dfrac{\alpha}{\beta \,\&\, \gamma} \qquad \text{and} \qquad \langle\beta\rangle \Rightarrow \delta \quad \dfrac{\beta}{\delta}$$

by derivations
$$\begin{array}{c}\alpha \\ - \\ - \\ - \\ \beta \,\&\, \gamma\end{array} \qquad \text{and} \qquad \begin{array}{c}\beta \\ + \\ + \\ + \\ \delta\end{array}$$

uniformly yields the required derivation, no matter how $\begin{smallmatrix}-\\-\\-\end{smallmatrix}$ and $\begin{smallmatrix}+\\+\\+\end{smallmatrix}$ look.

The distinction between admissibility and derivability breaks down in favour of derivability, however, if one allows additional assumptions from the beginning: If we call $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ admissible if there is a procedure which transforms derivations of α_i *from* Δ $(1 \leq i \leq n)$ into a derivation of α *from* Δ, then by taking Δ to be the system $\alpha_1, \ldots, \alpha_n$ we obtain the derivability of $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ from its admissibility. And similarly, if we call $\langle\langle\Gamma_1\rangle \Rightarrow \alpha_1, \ldots, \langle\Gamma_n\rangle \Rightarrow \alpha_n\rangle \Rightarrow \alpha$ admissible, if there is a procedure which transforms derivations of α_i from Γ_i *and* Φ $(1 \leq i \leq n)$ into a derivation of α *from* Φ, then by taking Φ to be the system $\langle\Gamma_1\rangle \Rightarrow \alpha_1, \ldots, \langle\Gamma_n\rangle \Rightarrow \alpha_n$ so that there are trivial derivations of α_i from Γ_i and Φ $(1 \leq i \leq n)$, we obtain the derivability of the considered rule from its admissibility.

In the following we shall rely on the notion of uniform procedures and thus on assumption rules not only when dealing with derivability in intuitionistic logic but also when dealing with the validity of rules. That means that we may use assumption rules throughout so that elimination procedures are, according to the last paragraph, eo ipso uniform ones.

2.2. Derivability - Canonical Conditions and Consequences

The explication of the notion of assumption rules leads to the following formal definitions of derivation and derivability which correspond to those one can find in Prawitz (1965). Let K be one of the calculi defined above. We consider formulas of K and rules which are built up from formulas of K. Derivations of formulas depending on finite sets of assumptions are defined as follows:

$\overline{\alpha}$ is a derivation of α in K depending on the empty set, if $\Rightarrow \alpha$ is a basic rule of K (i.e., an axiom), and depending on $\{\alpha\}$ otherwise.

If for each i $(1 \leq i \leq n)$,

$$\begin{array}{c}\overline{\Delta_i}\\ \vdots \\ \alpha_i\end{array}$$

is a derivation of α_i in K depending on M_i, then

$$\dfrac{\begin{array}{c}\overline{\Delta_1}^{\,(\times)}\\ \vdots \\ \alpha_1\end{array} \quad \cdots \quad \begin{array}{c}\overline{\Delta_n}^{\,(\times)}\\ \vdots \\ \alpha_n\end{array}}{\alpha}\,(\times)$$

is a derivation of α in K depending on $\bigcup_{i=1}^{n}(M_i \setminus \{\Delta_i\})$, if $\langle\langle\Delta_1\rangle \Rightarrow \alpha_1, \ldots, \langle\Delta_n\rangle \Rightarrow \alpha_n\rangle \Rightarrow \alpha$ is a basic rule of K, $\times$ is a numeral which has not been used in the derivation and $\overline{\Delta_i}^{(\times)}$ means that '($\times$)' is attached to all $\overline{\beta}$ where β is a topmost occurrence of a formula belonging to Δ_i.

If for each i ($1 \leq i \leq n$),

$$\begin{array}{c}\vdots\\ \alpha_i\end{array}$$

is a derivation of α_i in K depending on M_i, then

$$\dfrac{\begin{array}{ccc}\vdots & & \vdots\\ \alpha_1 & \ldots & \alpha_n\end{array}}{\alpha}$$

is a derivation of α in K depending on $(\bigcup_{i=1}^{n} M_i) \cup \{\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha\}$.

The derivations of α_i are called the *immediate subderivations* of the resulting derivation.

If $M \subseteq \{\Phi\}$, then a derivation of α *depending* on M in K is also called a derivation of α *from* Φ in K. α is derivable from Φ in K ($\Phi \vdash_K \alpha$) if there is a derivation of α from Φ in K. A rule ρ is derivable from Φ in K ($\Phi \vdash_K \rho$) iff $\Phi, (\rho)_1 \vdash_K (\rho)_2$. It is derivable in K ($\vdash_K \rho$) iff $(\rho)_1 \vdash_K (\rho)_2$. A system of formulas or rules Ψ is derivable from Φ in K ($\Phi \vdash_K \Psi$) if all its members are derivable from Φ in K.

The definition of the derivability of rules is justified by the remarks in section 2.1 above, according to which the assumption that a derivation of β from Δ is given can be represented by the assumption rule $\langle\Delta\rangle \Rightarrow \beta$ if this assumption is considered as an argument of *uniform* transformations only.

Following this definition the I rules can be conceived of as permitting the inference of formulas from systems of formulas and/or rules, namely of $\alpha \;\&\; \beta$ from the system α, β; of $\alpha \vee \beta$ from the systems α and β and of $\alpha \supset \beta$ from the system $\langle\alpha\rangle \Rightarrow \beta$. $\bot$ as a limiting case cannot be inferred from any system. The E rules can be conceived of as permitting the inference of formulas or rules from formulas, namely of α and β from $\alpha \;\&\; \beta$, of $\langle\langle\alpha\rangle \Rightarrow \gamma, \langle\beta\rangle \Rightarrow \gamma\rangle \Rightarrow \gamma$ for every γ from $\alpha \vee \beta$, of $\langle\alpha\rangle \Rightarrow \beta$ from $\alpha \supset \beta$ and of α for every α from $\bot$. This relationship between formulas and (systems of) rules leads to the following definition: Let α be a formula over Å. Then we define the predicate of being an α^-, which is a predicate for systems of formulas or rules, as follows:

α is an α^- if α is atomic.
β, γ is an α^- if α equals $\beta \;\&\; \gamma$.
β is an α^- if α equals $\beta \vee \gamma$.
γ is an α^- if α equals $\beta \vee \gamma$.
$\langle\beta\rangle \Rightarrow \gamma$ is an α^- if α equals $\beta \supset \gamma$.
There is no $\bot^-$ (not even the empty system!).

For a rule ρ, we define ρ itself to be a ρ^-, and for systems of formulas and/or rules Φ, Φ^- is defined memberwise, i.e., $\Psi_1, \ldots, \Psi_n$ is a Φ^-, if each Ψ_i is a Φ_i^- $(1 \leq i \leq n)$ and Φ equals $\Phi_1, \ldots, \Phi_n$ where each Φ_i $(1 \leq i \leq n)$ is either a formula or a rule. If Φ is empty, Φ is a Φ^-.

For given α, the α^- can be considered as the possibilities from which α can be introduced by application of an I rule. They will also be called the *canonical conditions* for α, and similarly for Φ^- and Φ.

Conversely, we define the predicate of being an α^+, which is a predicate for formulas or rules, as follows:

α is an α^+ if α is atomic.
β is an α^+ if α equals $\beta \,\&\, \gamma$.
γ is an α^+ if α equals $\beta \,\&\, \gamma$.
For each δ, $<<\beta> \Rightarrow \delta, <\gamma> \Rightarrow \delta> \Rightarrow \delta$ is an α^+ if α equals $\beta \vee \gamma$.
$<\beta> \Rightarrow \gamma$ is an α^+ if α equals $\beta \supset \gamma$.
For each γ, γ is an $\bot^+$.

Each α^+ can be considered as a possible consequence of α, if α is used as the major premiss of an E rule. The α^+ will also be called the *canonical consequences* of α.

In the following we shall use 'α^+', 'β^+', ..., 'α^-', 'β^-', ..., 'ρ^-', 'Φ^-', ..., as schematic letters for formulas, rules or systems of formulas and/or rules, which are α^+, β^+, ..., α^-, β^-, ..., ρ^-, Φ^-, ... That is, we also use these metalinguistic predicate signs as schematic letters for what falls under the predicates.

Lemma 2.2.1: (i) From each Φ^-, Φ can be inferred by use of I rules for formulas which are members of Φ.

(ii) Each α^+ can be inferred from α by use of an E rule for α.

Proof: (i) Let Φ be the system $\Phi_1, \ldots, \Phi_n$ where each Φ_i $(1 \leq i \leq n)$ is either a formula or a rule. We show that for each i $(1 \leq i \leq n)$, each Φ_i^- implies Φ_i in the described way. If Φ_i is a rule, then each Φ_i^- is identical with Φ_i. Let Φ_i be α. Then the assertion follows from the fact that for all α^-, α follows from α^- by use of the I rule for α. (If α is $\bot$, there is no α^-, so '*for all* α^- ...' is vacuously fulfilled.)

(ii) Immediately from the definition of α^+ and the E rules for α.

Lemma 2.2.2: (i) For each α^- and each α^+: α^+ can be derived from α^- without use of any basic rule.

(ii) For each α there are α^+ such that α can be derived with the help of these α^+ by use of the I rule for α (if α is non-atomic).

Proof: (i) If α is atomic, nothing remains to be shown. If α is β & γ, then α^+ is either β or γ and α^- is the system β, γ. Now both β and γ are derivable from β, γ without a basic rule. If α is $\beta \vee \gamma$, then α^- is either β or γ, and each α^+ is of the form $\langle\langle\beta\rangle \Rightarrow \delta, \langle\gamma\rangle \Rightarrow \delta\rangle \Rightarrow \delta$. Now δ follows both from $\beta, \langle\beta\rangle \Rightarrow \delta, \langle\gamma\rangle \Rightarrow \delta$ and from $\gamma, \langle\beta\rangle \Rightarrow \delta, \langle\gamma\rangle \Rightarrow \delta$ without application of a basic rule. If α is $\beta \supset \gamma$, then both α^+ and α^- equal $\langle\beta\rangle \Rightarrow \gamma$, and γ trivially follows from $\beta, \langle\beta\rangle \Rightarrow \gamma$ without application of a basic rule. If α is $\bot$, the assertion is vacuously fulfilled, because there is no $\bot^-$.

(ii) If α is β & γ, then both β and γ are α^+, and β & γ follows from β, γ by the I rule for α. If α is $\beta \vee \gamma$, consider $\langle\langle\beta\rangle \Rightarrow \beta \vee \gamma, \langle\gamma\rangle \Rightarrow \beta \vee \gamma\rangle \Rightarrow \beta \vee \gamma$ which is an α^+. With the help of this rule, $\beta \vee \gamma$ can be obtained, since $\beta \vee \gamma$ follows both from β and from γ by the I rule for α. If α is $\beta \supset \gamma$, then each α^+ equals to $\langle\beta\rangle \Rightarrow \gamma$, and $\beta \supset \gamma$ follows from $\langle\beta\rangle \Rightarrow \gamma$ by use of the I rule for α. If α is $\bot$, then $\bot$ itself is an $\bot^+$.

(i) shows that for all α^- and α^+ the way from α^- to α^+ via α is a detour and can be avoided by directly moving from α^- to α^+. This is closely related to the fact that adding I and E rules for logical operators to a calculus forms a conservative extension of that calculus. (ii) shows that I and E rules uniquely determine logical constants: For if α and β are of such a kind that the sets of all α^- and of all β^- are identical as well as of all α^+ and of all β^+ (i.e., α and β have the same canonical conditions and consequences), then we may pass from β by E rules to all α^+, and from some of these α^+ to α by I rules (according to (ii)), and vice versa. I.e., α is uniquely determined by the sets of canonical conditions and canonical consequences of α. (For a general investigation of conservativeness and uniqueness in relation to conditions and consequences of sentences see Došen and Schroeder-Heister, 1984.)

2.3. Validity and Harmony

The concepts of Φ^- and α^+ can now be used to redefine the notions of validity+ and validity- respectively, of a rule $\langle\Phi\rangle \Rightarrow \alpha$. The idea is roughly as follows: instead of saying that applications of $\langle\Phi\rangle \Rightarrow \alpha$ can be eliminated from derivations in which the principal premises of applications of this rule are always conclusions of applications of I rules, we could simply formulate that α is derivable from all Φ^- without that rule. For if $\langle\Phi\rangle \Rightarrow \alpha$ can be eliminated from the described derivations, it can be eliminated from the derivation of α from all Φ^- which infers Φ from Φ^- and then applies $\langle\Phi\rangle \Rightarrow \alpha$; and if α is derivable from all Φ^-, the eliminability of $\langle\Phi\rangle \Rightarrow \alpha$ follows because, according to the restrictions on the considered derivations, applications of $\langle\Phi\rangle \Rightarrow \alpha$ presuppose a derivation of a Φ^- and can be evaded by immediately deriving α from this Φ^-. Similarly, instead of saying that applications of $\langle\Phi\rangle \Rightarrow \alpha$ can be eliminated from derivations in which the conclusion α of such applications is always a major premiss of an E rule, we could simply formulate that all α^+ are derivable from Φ without that rule. For if

$<\Phi> \Rightarrow \alpha$ can be eliminated from the described derivations it can be eliminated from all derivations of α^+ from Φ which apply $<\Phi> \Rightarrow \alpha$ and then infer α^+ from α; and if all α^+ are derivable from Φ the eliminability of $<\Phi> \Rightarrow \alpha$ follows because, according to the restrictions on the considered derivations, applications of $<\Phi> \Rightarrow \alpha$ are followed by inferring an α^+ in the next step and can be evaded by immediately deriving this α^+ from Φ.

The resulting definitions of validity - according to which, roughly speaking, $<\Phi> \Rightarrow \alpha$ is valid+ if α is derivable from all Φ^- by means of I rules and valid+ rules of lower complexity, and valid- if all α^+ are derivable from Φ by means of E rules and valid- rules of lower complexity (for exact definitions see the next section) - can be programmatically stated as follows, using the terms 'canonical condition' and 'canonical consequence' and neglecting the inductive character of the definitions of validity: A rule is valid+ if its conclusion follows from all canonical conditions of its premisses, and is valid- if from its premisses all canonical consequences of its conclusion follow. In other words, the definitions of validity guarantee that everything that follows from all canonical conditions of a system of formulas and/or rules follows from that system itself, and everything that implies all canonical consequences of a formula implies that formula itself.

This can be put in still another way: Let us call α stronger+ than β if β is derivable from α by means of I rules and valid+ rules, and weaker- than β if α is derivable from β by means of E rules and valid- rules. α is called a strongest+ formula fulfilling a given condition if α is stronger+ than all other formulas β fulfilling this condition. α is called a weakest- formula fulfilling a given condition if α is weaker- than all other formulas β fulfilling this condition. Then each α is a strongest formula which can be inferred from all α^- by I rules and valid+ rules: For if β can be derived from all α^- by such rules, then $\langle\alpha\rangle \Rightarrow \beta$ is valid+; thus β is derivable from α by means of I rules and valid+ rules. The inductive definition of validity+ can be viewed as a stepwise addition of valid+ rules in order to guarantee this maximal strength of conclusions of I rules. Similarly, each α is a weakest- formula from which each α^+ can be inferred by E rules and valid- rules: For if each α^+ can be derived from β by such rules, then $\langle\beta\rangle \Rightarrow \alpha$ is valid-; thus α is derivable from β by means of E rules and valid- rules. Again, the inductive definition of validity- adds on each level valid- rules in order to guarantee this minimal strength of major premisses of E rules.

This formulation is very close to what Tennant (1978) calls the Principle of Harmony: "Introduction and elimination rules for a logical operator λ must be formulated so that a sentence with λ dominant expresses the *strongest* proposition which can be inferred from the stated premisses when the conditions for λ-introduction are satisfied; while it expresses the *weakest* proposition which can feature in the way required for λ-elimination." (p. 74)

It should be noted, however, that though the principle of harmony follows from our definitions of validity, the fact that the principle of harmony holds does not guarantee validity. As can easily be checked, the principle of harmony is not affected if one adds unvalid+ E rules as, e.g.,

$$\frac{\alpha \ \& \ \beta}{\gamma} \ .$$

Concerning these nice symmetries between the two approaches to validity one should, however, be aware that only the notion of validity- is adequate for the full system of intuitionistic logic.

2.4. The Definitions of Validity - Basic Lemmata

Let an atomic calculus A be given. We assume that the considered formulas and rules are formulas and rules over A. Without always mentioning it, we assume in all contexts of validity+, $I^{o}(A)$ and $I^{+}(A)$, that the formulas, rules and systems of formulas or rules do not contain $\vee$. Since the definitions of the two validity concepts run completely parallel we give them at the same time, presenting the different formulations for the second notion in square brackets.

A *derivation* is called *k-valid+* [*k-valid-*] in A if it is a derivation in the calculus which results from $I^{+}(A)$ [$I^{-}(A)$] by adding all rules of rank $\leq$ k which are valid+ [valid-] in A. If there is a derivation of α from Φ which is k-valid+ [k-valid-] in A, we write $\Phi \vdash^{A,\,+}_{k} \alpha$ [$\Phi \vdash^{A,\,-}_{k} \alpha$]. A *rule* $\langle\Phi\rangle \Rightarrow \alpha$ is called valid+ [valid-] in A if for all Φ^{-} [α^{+}], $\Phi^{-} \vdash^{A,\,+}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha$ [$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha^{+}$].

Obviously, rules of rank 1 are valid+ [valid-] if they are derivable in $I^{+}(A)$ [$I^{-}(A)$] and therefore in A (because their premises and conclusions must be atomic).

An *interpretation in* A is a function which assigns a formula of A to each sentence letter, i.e., a function from the formulas of F into the formulas of A. If ρ is a rule over F (i.e., a rule of I) and φ an interpretation in A, then ρ^{φ} is the rule over A which results from ρ by replacing sentence letters a by $\varphi(a)$ in ρ. Obviously, $|\rho| = |\rho^{\varphi}|$. A rule of I^{o} [I] is called valid+ [valid-] if for each atomic calculus A and for each interpretation φ in A, ρ^{φ} is valid+ [valid-] in A.

Lemma 2.4.1: A rule ρ of I^{o} [I] is valid+ [valid-] iff it is valid+ [valid-] in F.

Proof: The direction from left to right is trivial because the identity is an interpretation in F. For the direction from right to left we use induction on the rank of ρ. Let φ be an interpretation in A. If ρ is of the form $\langle\Phi\rangle \Rightarrow \alpha$ and valid+ [valid-] in F, then for all Φ^- [α^+], $\Phi^- \vdash^{F,\,+}_{|\rho|\,-\,1} \alpha$ [$\Phi \vdash^{F,\,-}_{|\rho|\,-\,1} \alpha^+$]. Since all basic rules of I^+ [I^-] become basic rules of $I^+(A)$ [$I^-(A)$] when interpreted in A, and the same holds by induction hypothesis for rules of rank $< |\rho|$ which are valid+ [valid-] in A, we have $(\Phi^\varphi)^- \vdash^{A,\,+}_{|\rho|\,-\,1} \alpha^\varphi$ [$\Phi^\varphi \vdash^{A,\,-}_{|\rho|\,-\,1} (\alpha^\varphi)^+$], i.e., ρ^φ is valid+ [valid-] in A.

Lemma 2.4.2: If $\Phi \vdash^{A,\,+}_{k} \alpha$ [$\Phi \vdash^{A,\,-}_{k} \alpha$], then $\Phi \vdash^{A,\,+}_{l} \alpha$ [$\Phi \vdash^{A,\,-}_{l} \alpha$] for every $l > k$.

Proof: trivial.

Lemma 2.4.3: If $\langle\Phi\rangle \Rightarrow \alpha$ is valid+ [valid-] in A, then $\langle\Phi, \Psi\rangle \Rightarrow \alpha$ is valid+ [valid-] in A.

Proof: trivial.

Lemma 2.4.4: If $\langle\Phi\rangle \Rightarrow \alpha$ is valid+ [valid-] in A, then $\Phi \vdash^{A,\,+}_{|\langle\Phi\rangle \Rightarrow \alpha|} \alpha$ [$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \alpha|} \alpha$].

Proof: trivial.

Lemma 2.4.5: (i) All basic rules of $I^{\circ}(A)$ are valid+ in A.

(ii) All basic rules of $I(A)$ are valid- in A.

Proof: (i) For the basic rules of A nothing remains to be shown. For the I rules the assertion follows by twofold application of lemma 2.2.1(i). For the E rules the assertion follows from lemma 2.2.2(i) where in the case of $\supset$E lemma 2.2.1(i) is still to be applied to the minor premiss.

(ii) For the basic rules of A nothing remains to be shown. For the E rules the assertion follows by twofold application of lemma 2.2.1(ii). For the I rules the assertion follows from lemma 2.2.2(i).

Lemma 2.4.6: If $<\Phi> \Rightarrow \alpha$ is valid+ in A and α is different from $\perp$ then for each Φ^- there is an α^- such that $\Phi^- \vdash^{A,\ +}_{|<\Phi> \Rightarrow \alpha| - 1} \alpha^-$.

Proof: By assumption, we have a derivation D of α from Φ^- which is k-valid+ in A for $k = |<\Phi> \Rightarrow \alpha| - 1$. If α is atomic, then all α^- are equal to α and nothing remains to be shown. Let α be non-atomic. If $|\Phi| > |\alpha|$, then by lemma 2.4.5(i) and 2.1.1(i) we obtain a derivation of any α^- by application of an E rule to α which is k-valid+ (note that α is not a disjunction). If $|\Phi| \leq |\alpha|$ then $|<\Phi> \Rightarrow \alpha| = |\alpha| + 1$. Thus D is $|\alpha|$-valid+ in A. In the last step D cannot apply an assumption rule $<\Psi> \Rightarrow \alpha$ belonging to Φ^-, because $|\Phi^-| \leq |\Phi| \leq |\alpha|$, but $|<\Psi> \Rightarrow \alpha| > |\alpha|$. Likewise, D cannot apply a rule of rank $\leq |\alpha|$ which is valid+ in A. α cannot even be an assumption formula of D, for then α would belong to a β^- for a member β of Φ, and because α is a formula, $|\alpha| < |\beta|$ would hold, contradicting $|\beta| \leq |\Phi| \leq |\alpha|$ ($|\beta^-| = |\beta|$ only holds if β is an implication and β^- a rule). So D must apply an I rule in the last step.

Lemma 2.4.7: (i) If for all α^-, Φ, $\alpha^- \vdash^{A,\ +}_{k} \beta$, then Φ, $\alpha \vdash^{A,\ +}_{l} \beta$, where $l = \max(k, |\alpha| + 1)$.

(ii) If for all Φ^-, $\Phi^- \vdash_{I^\circ(A)} \alpha$, then $\Phi \vdash_{I^\circ(A)} \alpha$.

Proof: (i) If α^- is identical with α, nothing remains to be shown. If α is a conjunction or implication, apply the E rule for α and use lemma 2.1.1(i). If α is $\perp$, then by applications of $\perp$E rules with atomic conclusions (which are thus of rank $|\alpha| + 1$) we can obtain all atomic subformulas and subformulas of the form $\perp$ of β.

Therefrom β is derivable by use of I rules.

(ii) Similarly.

For the calculus $I(A)$ (and therefore for $I^{o}(A)$) a normalization theorem in the sense of Prawitz (1965) can be proved. Because of the possible occurrence of assumption rules certain modifications are necessary (cf. Schroeder-Heister, 1981) but the central conclusions like the subformula property can be upheld. Without proof we state what we shall need.

Lemma 2.4.8: Let a derivation D of α from Φ in $I(A)$ or $I^{o}(A)$ be given. It can be transformed into a normal derivation D' of α from Φ in $I(A)$ or $I^{o}(A)$, respectively, for which the following holds: if D' applies an E rule in the last step with β as major premiss, then β is a subformula of a formula occurring in Φ and $|\beta| \leq |\Phi|$. Furthermore all applications of $\bot$E rules have atomic conclusions.

3. SOUNDNESS AND COMPLETENESS OF INTUITIONISTIC LOGIC WITH RESPECT TO VALIDITY-

Let an arbitrary atomic calulus A be given.

Theorem 3.1 (Soundness): If $\Phi \vdash_{I(A)} \alpha$, then $\langle\Phi\rangle \Rightarrow \alpha$ is valid- in A. (I.e., each rule which is derivable in $I(A)$ is valid- in A.)

Proof: By lemma 2.4.8 we can assume that derivations in $I(A)$ are in normal form. We proceed by induction on the length of normal derivations. We have to distinguish five main cases according to the kinds of rules which might have been applied in the last step of the given normal derivation D of α from Φ (cf. the clauses in the definition of a derivation).

(i) If D is of the form $\overline{\alpha}$ where $\Rightarrow \alpha$ is a basic rule of $I(A)$, then $\Rightarrow \alpha$ is a basic rule of A (because there are no premiss-free I or E rules), thus $\vdash^{A,\,-}_{0} \alpha$, therefore $\langle\Phi\rangle \Rightarrow \alpha$ is valid- in A.

(ii) If D is of the form $\overline{\alpha}$ where α is an assumption (thus belonging to Φ), then for all α^+, $\alpha \vdash^{A,\,-}_{0} \alpha^+$ holds by lemma 2.2.1(ii), therefore $\langle\Phi\rangle \Rightarrow \alpha$ is valid- in A.

(iii) If D applies an I rule in the last step, we distinguish subcases according to the form of α.

a) If α is $\alpha_1 \,\&\, \alpha_2$, then D contains derivations of α_1 and of α_2 from Φ as immediate subderivations. Thus by induction hypothesis $\langle\Phi\rangle \Rightarrow \alpha_1$ and $\langle\Phi\rangle \Rightarrow \alpha_2$ are valid- in A, i.e., $\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \alpha_i| - 1} \alpha_i^+$ for all α_i^+ $(i = 1, 2)$.

Thus by lemma 2.2.2(ii) and lemma 2.1.1(i):

$$\Phi \vdash^{A,\,-}_{k_i} \alpha_i \quad (i = 1, 2)$$

where $k_i = \max(|\langle\Phi\rangle \Rightarrow \alpha_i| - 1,\ |\alpha_i| + 1)$.

Thus by lemma 2.1.1(ii):

$$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha_i \quad (i = 1, 2).$$

b) If α is $\alpha_1 \vee \alpha_2$, then D contains a derivation of α_1 or of α_2 from Φ as an immediate subderivation. Thus by induction hypothesis $\langle\Phi\rangle \Rightarrow \alpha_i$ is valid- in A for some i $(i = 1, 2)$. As in a) we obtain

$$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha_i.$$ Thus

$$\Phi,\ \langle\alpha_1\rangle \Rightarrow \beta,\ \langle\alpha_2\rangle \Rightarrow \beta \vdash^{A,\ -}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \beta \text{ for each } \beta.$$

c) If α is $\alpha_1 \supset \alpha_2$, then D contains a derivation of α_2 from Φ, α_1 as an immediate subderivation. Thus by induction hypothesis $\langle\Phi,\ \alpha_1\rangle \Rightarrow \alpha_2$ is valid- in A. As before we obtain

$$\Phi,\ \alpha_1 \vdash^{A,\ -}_{|\langle\Phi,\ \alpha_1\rangle \Rightarrow \alpha| - 1} \alpha_2$$

where $|\langle\Phi,\ \alpha_1\rangle \Rightarrow \alpha| = |\langle\Phi\rangle \Rightarrow \alpha|$.

(iv) If D applies an E rule in the last step, we distinguish subcases according to the form of its major premiss δ.

a) If δ is $\alpha\ \&\ \beta$ or $\beta\ \&\ \alpha$, then D contains a derivation of $\alpha\ \&\ \beta$ or $\beta\ \&\ \alpha$, respectively, from Φ as an immediate subderivation. By induction hypothesis $\langle\Phi\rangle \Rightarrow \alpha\ \&\ \beta$ or $\langle\Phi\rangle \Rightarrow \beta\ \&\ \alpha$, respectively, is valid- in A, i.e., in particular

$$\Phi \vdash^{A,\ -}_{|\langle\Phi\rangle \Rightarrow \alpha\ \&\ \beta| - 1} \alpha.$$

Thus by lemma 2.2.1(ii):

$$\Phi \vdash^{A,\ -}_{|\langle\Phi\rangle \Rightarrow \alpha\ \&\ \beta| - 1} \alpha^+ \text{ for each } \alpha^+.$$

By lemma 2.4.8: $|\Phi| \geq |\alpha\ \&\ \beta|$, thus $|\langle\Phi\rangle \Rightarrow \alpha\ \&\ \beta| = |\Phi| + 1 =$ $= |\langle\Phi\rangle \Rightarrow \alpha|$.

b) If δ is $\beta \vee \gamma$, then D contains derivations of $\beta \vee \gamma$ from Φ, of α from Φ, β and of α from Φ, γ as immediate subderivations. By induction hypothesis $\langle\Phi\rangle \Rightarrow \beta \vee \gamma$, $\langle\Phi,\ \beta\rangle \Rightarrow \alpha$ and $\langle\Phi,\ \gamma\rangle \Rightarrow \alpha$ are valid- in A, i.e.,

$$\Phi,\ \langle\beta\rangle \Rightarrow \varepsilon,\ \langle\gamma\rangle \Rightarrow \varepsilon \mathrel{\mathop{\vdash}\limits^{A,\ -}_{|\langle\Phi\rangle \Rightarrow \beta \vee \gamma| - 1}} \varepsilon \text{ for all } \varepsilon$$

$$\Phi,\ \beta \mathrel{\mathop{\vdash}\limits^{A,\ -}_{|\langle\Phi,\ \beta\rangle \Rightarrow \alpha| - 1}} \alpha^{+} \text{ for all } \alpha^{+}$$

$$\Phi,\ \gamma \mathrel{\mathop{\vdash}\limits^{A,\ -}_{|\langle\Phi,\ \gamma\rangle \Rightarrow \alpha| - 1}} \alpha^{+} \text{ for all } \alpha^{+}.$$

By specializing ε to α^{+} if α^{+} is a formula and to $(\alpha^{+})_2$ if α^{+} is a rule, we obtain $\Phi \mathrel{\mathop{\vdash}\limits^{A,\ -}_{k - 1}} \alpha^{+}$ for all α^{+}, where $k = \max(|\langle\Phi\rangle \Rightarrow \beta \vee \gamma|, |\langle\Phi, \beta\rangle \Rightarrow \alpha|, |\langle\Phi, \gamma\rangle \Rightarrow \alpha|)$. By lemma 2.4.8: $|\Phi| \geq |\beta \vee \gamma|$, thus $|\langle\Phi\rangle \Rightarrow \beta \vee \gamma| = |\Phi| + 1 = |\Phi, \beta| + 1 = |\Phi, \gamma| + 1 \leq |\langle\Phi\rangle \Rightarrow \alpha|$.

c) If δ is $\beta \supset \alpha$ then D contains derivations of $\beta \supset \alpha$ from Φ and of β from Φ as immediate subderivations. By induction hypothesis $\langle\Phi\rangle \Rightarrow \beta \supset \alpha$ and $\langle\Phi\rangle \Rightarrow \beta$ are valid- in A, i.e.,

$$\Phi,\ \beta \mathrel{\mathop{\vdash}\limits^{A,\ -}_{|\langle\Phi\rangle \Rightarrow \beta \supset \alpha| - 1}} \alpha$$

and

$$\Phi \mathrel{\mathop{\vdash}\limits^{A,\ -}_{|\langle\Phi\rangle \Rightarrow \beta| - 1}} \beta^{+} \text{ for all } \beta^{+}.$$

Thus by lemma 2.2.2(ii) and lemma 2.1.1(i):

$$\Phi \mathrel{\mathop{\vdash}\limits^{A,\ -}_{k}} \beta$$

where $k = \max(|\langle\Phi\rangle \Rightarrow \beta| - 1, |\beta| + 1)$, thus by lemma 2.1.1(ii)

$$\Phi \mathrel{\mathop{\vdash}\limits^{A,\ -}_{|\langle\Phi\rangle \Rightarrow \beta \supset \alpha| - 1}} \beta, \text{ thus}$$

$$\Phi \mathrel{\mathop{\vdash}\limits^{A,\ -}_{|\langle\Phi\rangle \Rightarrow \beta \supset \alpha| - 1}} \alpha.$$

By lemma 2.2.1(ii):

$$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \beta \supset \alpha| - 1} \alpha^{+} \text{ for all } \alpha^{+}.$$

By lemma 2.4.8: $|\Phi| \geq |\beta \supset \alpha|$, thus $|\langle\Phi\rangle \Rightarrow \beta \supset \alpha| = |\Phi| + 1 =$ $= |\langle\Phi\rangle \Rightarrow \alpha|$.

d) If δ is $\bot$, then D contains a derivation of $\bot$ from Φ as an immediate subderivation. By induction hypothesis $\langle\Phi\rangle \Rightarrow \bot$ is valid- in A, i.e., in particular

$$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \bot| - 1} \alpha.$$

By lemma 2.4.8: $|\Phi| \geq \bot$, thus $|\langle\Phi\rangle \Rightarrow \bot| = |\Phi| + 1 \leq |\langle\Phi\rangle \Rightarrow \alpha|$. Furthermore α is atomic, i.e., each α^{+} is identical with α.

(v) D applies an assumption rule $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ belonging to Φ in the last step. Then for each i $(1 \leq i \leq n)$ D contains derivations of α_i from Φ as immediate subderivations. By induction hypothesis, for each i $(1 \leq i \leq n)$, $\langle\Phi\rangle \Rightarrow \alpha_i$ is valid- in A, i.e.,

$$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \alpha_i| - 1} \alpha_i^{+} \text{ for each } \alpha_i^{+}.$$

Thus by lemma 2.2.2(ii) and lemma 2.1.1(i)

$$\Phi \vdash^{A,\,-}_{k_i} \alpha_i \text{ for all i } (1 \leq i \leq n)$$

where $k_i = \max(|\langle\Phi\rangle \Rightarrow \alpha_i| - 1,\ |\alpha_i| + 1)$. Because $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ belongs to Φ, we have $|\alpha| < |\Phi|$ and $|\alpha_i| < |\Phi|$ for all i $(1 \leq i \leq n)$. Thus $k_i = |\Phi| = |\langle\Phi\rangle \Rightarrow \alpha| - 1$ for every i $(1 \leq i \leq n)$. Thus

$$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha$$

by application of $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ (which belongs to Φ), thus by lemma 2.2.1(ii):

$$\Phi \vdash^{A,\,-}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha^+ \text{ for each } \alpha^+.$$

Theorem 3.2 (Completeness): For any k, if $\Phi \vdash^{A,\,-}_{k} \alpha$ then $\Phi \vdash_{I(A)} \alpha$.

Proof: by induction on the pair consisting of k and the length of derivations D of α from Φ which are k-valid- in A. If D is 0-valid- in A, nothing remains to be shown. Let D be k-valid- in A for k > 0.

(i) If D is of the form $\overline{\alpha}$ where $\Rightarrow \alpha$ is a basic rule of $I^-(A)$, then α is a basic rule of A, so D is already 0-valid- in A. If $\Rightarrow \alpha$ is a rule which is valid- in A and for which $|\Rightarrow \alpha| \leq k$, then $\vdash^{A,\,-}_{k-1} \alpha^+$ for all α^+. So by induction hypothesis $\vdash_{I(A)} \alpha^+$ for all α^+, therefore $\vdash_{I(A)} \alpha$ by lemma 2.2.2(ii).

(ii) If D is of the form $\overline{\alpha}$ where α is an assumption (thus belonging to Φ), then $\Phi \vdash_{I(A)} \alpha$ holds trivially.

(iii) If D applies a rule ρ of the form $\langle\langle\Gamma_1\rangle \Rightarrow \alpha_1, \ldots, \langle\Gamma_n\rangle \Rightarrow \alpha_n\rangle \Rightarrow \alpha$ in the last step, then D contains for all i ($1 \leq i \leq n$) derivations of α_i from Γ_i, Φ as immediate subderivations which are k-valid- in A. By induction hypothesis, we have

(5) $\Gamma_i, \Phi \vdash_{I(A)} \alpha_i$.

We distinguish three subcases:

a) If ρ is valid- in A and $|\rho| \leq k$, then $(\rho)_1 \vdash^{A,\,-}_{k-1} \alpha^+$ for all α^+. So by induction hypothesis $(\rho)_1 \vdash_{I(A)} \alpha^+$, therefore $(\rho)_1 \vdash_{I(A)} \alpha$ by lemma 2.2.2(ii). Together with (5) we obtain $\Phi \vdash_{I(A)} \alpha$.

b) If ρ is a basic rule of $I^-(A)$ then ρ is also a basic rule of $I(A)$ and together with (5) we obtain $\Phi \vdash_{I(A)} \alpha$.

c) If ρ is an assumption rule then ρ belongs to Φ and together with (5) we obtain $\Phi \vdash_{I(A)} \alpha$.

Corollary 3.3: If $\langle\Phi\rangle \Rightarrow \alpha$ is valid- in A, then $\Phi \vdash_{I(A)} \alpha$ (i.e., each rule which is valid- in A is derivable in $I(A)$).

Proof: From lemma 2.4.4 and theorem 3.2.

Corollary 3.4: If $\Phi \vdash^{A,\,-}_{k} \alpha$ for some k, then $\langle\Phi\rangle \Rightarrow \alpha$ is valid- in A.

Proof: Theorems 3.2 and 3.1.

This corollary shows that validity- of rules is transitive in the sense that rules which have been derived by successive application of valid- rules are themselves valid-.

Corollary 3.5: A rule over F is derivable in I iff it is valid-.

Proof: Lemma 2.4.1.

This shows that intuitionistic logic with sentence letters is also 'formally' sound and complete with respect to validity-, i.e., sound and complete under all interpretations in atomic calculi.

4. SOUNDNESS AND COMPLETENESS WITH RESPECT TO VALIDITY+

We consider an atomic calculus A and formulas over A which do not contain $\vee$.

Theorem 4.1 (Soundness): If $\Phi \vdash_{I^{o}(A)} \alpha$, then $\langle\Phi\rangle \Rightarrow \alpha$ is valid+ in A.

Proof: We proceed as in the proof of theorem 3.1, assuming that the given derivation D of α from Φ in $I^{o}(A)$ is in normal form. The case (i) is treated analogously.

(ii) If D is of the form $\overline{\alpha}$ where α is an assumption (thus belonging to Φ), then $\alpha^{-} \vdash^{A,\ +}_{0} \alpha$ holds according to lemma 2.2.1(i), therefore $\langle\Phi\rangle \Rightarrow \alpha$ is valid+ in A.

(iii) If D applies an I rule in the last step we distinguish subcases according to the form of α.

a) If α is α_1 & α_2, then by induction hypothesis $\langle\Phi\rangle \Rightarrow \alpha_1$ and $\langle\Phi\rangle \Rightarrow \alpha_2$ are valid+ in A, i.e., for all Φ^{-}:

$$\Phi^{-} \vdash^{A,\ +}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha_i \quad (i = 1, 2).$$

Thus by application of an &I rule (which is at our disposal in $I^{+}(A)$):

$$\Phi^{-} \vdash^{A,\ +}_{|\langle\Phi\rangle \Rightarrow \alpha_i| - 1} \alpha \text{ for all } \Phi^{-}.$$

b) If α is $\alpha_1 \supset \alpha_2$, then by induction hypothesis $\langle\Phi, \alpha_1\rangle \Rightarrow \alpha_2$ is valid+ in A, i.e., for all Φ^{-}, α_1^{-}:

$$\Phi^-, \quad \alpha_1^- \vdash^{A, +}_{|<\Phi, \alpha_1> \Rightarrow \alpha_2| - 1} \alpha_2,$$ thus by lemma 2.4.7(i)

$$\Phi^-, \quad \alpha_1 \vdash^{A, +}_{|<\Phi> \Rightarrow \alpha| - 1} \alpha_2$$ (note that $|\alpha_1| + 1 \leq |\alpha| < |<\Phi> \Rightarrow \alpha|$).

Thus by application of an $\supset$I rule

$$\Phi^- \vdash^{A, +}_{|<\Phi> \Rightarrow \alpha| - 1} \alpha.$$

(iv) If D applies an E rule in the last step, we distinguish subcases according to the form of its major premiss δ.

a) If δ is α & β, then by induction hypothesis $<\Phi> \Rightarrow \alpha \,\&\, \beta$ is valid+ in A. Thus by lemma 2.4.6

$$\Phi^- \vdash^{A, +}_{|<\Phi> \Rightarrow \alpha \& \beta| - 1} \alpha$$ for all Φ^-.

By lemma 2.4.8, $|\Phi| \geq |\alpha \,\&\, \beta| > |\alpha|$, thus $|<\Phi> \Rightarrow \alpha \,\&\, \beta| = |<\Phi> \Rightarrow \alpha|$.

b) If δ is $\beta \supset \alpha$, then by induction hypothesis $<\Phi> \Rightarrow \beta \supset \alpha$ and $<\Phi> \Rightarrow \beta$ are valid+ in A.

Therefore by lemma 2.4.6:

$$\Phi^-, \ \beta \vdash^{A, +}_{|<\Phi> \Rightarrow \alpha \supset \beta| - 1} \alpha$$ for all Φ^-.

Together with

$$\Phi^- \vdash^{A, +}_{|<\Phi> \Rightarrow \beta| - 1} \beta$$ for all Φ^-

we obtain

$$\Phi^- \vdash^{A,\ +}_{|\langle\Phi\rangle \Rightarrow \alpha \supset \beta| - 1} \alpha \text{ for all } \Phi^-$$

(because $|\langle\Phi\rangle \Rightarrow \alpha \supset \beta| \geq |\langle\Phi\rangle \Rightarrow \beta|$). By lemma 2.4.8, $|\Phi| \geq |\alpha \supset \beta| > |\alpha|$, thus $|\langle\Phi\rangle \Rightarrow \alpha \supset \beta| = |\langle\Phi\rangle \Rightarrow \alpha|$.

c) If δ is $\perp$, then by induction hypothesis $\langle\Phi\rangle \Rightarrow \perp$ is valid+ in A, i.e.,

$$\Phi^- \vdash^{A,\ +}_{|\langle\Phi\rangle \Rightarrow \perp| - 1} \perp \text{ for all } \Phi^-.$$

According to lemma 2.4.8, α is atomic, and $\perp$ is a subformula of a formula occurring in Φ. Thus $|\Phi| \geq 1$. If $|\Phi| > 1$, then $|\langle\Phi\rangle \Rightarrow \perp| = |\langle\Phi\rangle \Rightarrow \alpha| > 2$. Therefore application of the $\perp$E rule $\langle\perp\rangle \Rightarrow \alpha$, which is valid+ in A and for which $|\langle\perp\rangle \Rightarrow \alpha| = 2$ holds, yields

$$\Phi^- \vdash^{A,\ +}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha \text{ for all } \Phi^-.$$

If $|\Phi| = 1$, this is vacuously fulfilled, because $\perp$ then is a member of Φ.

(v) If D applies an assumption rule $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ in the last step, then $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ belongs to Φ and we have $|\Phi| > |\alpha_i|$ for all i ($1 \leq i \leq n$) and $|\Phi| > |\alpha|$, thus $|\langle\Phi\rangle \Rightarrow \alpha_i| = |\langle\Phi\rangle \Rightarrow \alpha| = |\Phi| + 1$ for all i ($1 \leq i \leq n$). By induction hypothesis for each i ($1 \leq i \leq n$), $\langle\Phi\rangle \Rightarrow \alpha_i$ is valid+ in A, i.e., for all Φ^-:

$$\Phi^- \vdash^{A,\ +}_{|\Phi|} \alpha_i.$$

Therefore

$$\Phi^-, \langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha \vdash^{A,\ +}_{|\Phi|} \alpha$$

which is the same as

$$\Phi^- \vdash^{A,\ +}_{|\langle\Phi\rangle \Rightarrow \alpha| - 1} \alpha$$

because $\langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \alpha$ is contained in all Φ^-.

Theorem 4.2 (Completeness): For any k, if $\Phi \vdash^{A,\ +}_{k} \alpha$, then $\Phi \vdash_{I^o(A)} \alpha$.

Proof: Analogously to the proof of theorem 3.2. We only mark the points where there are differences in argumentation.

(i) If $\Rightarrow \alpha$ is a rule which is valid+ in A and for which $|\Rightarrow \alpha| \leq k$, then $\vdash^{A,\ +}_{k-1} \alpha$. So by induction hypothesis $\vdash_{I^o(A)} \alpha$.

(iii) a) If D applies a rule $\langle\Psi\rangle \Rightarrow \alpha$ in the last step which is valid+ in A and whose rank is $\leq$ k, then by induction hypothesis (concerning the length of derivations)

(6) $\Phi \vdash_{I^o(A)} \Psi$.

Because of the validity+ of $\langle\Psi\rangle \Rightarrow \alpha$ in A we have

$$\Psi^- \vdash^{A,\ +}_{k-1} \alpha \text{ for all } \Psi^-,$$

so by induction hypothesis (concerning k)

$$\Psi^- \vdash_{I^o(A)} \alpha \text{ for all } \Psi^-,$$

therefore by lemma 2.4.7(ii)

$\Psi \vdash_{\overline{I^{o}(A)}} \alpha$.

Together with (6) $\Phi \vdash_{\overline{I^{o}(A)}} \alpha$ follows.

The corollaries 3.3 to 3.5 can easily be taken over.

5. SOME FURTHER POINTS

5.1 Extension to Quantifier Logic

We have restricted ourselves to sentential logic but an extension to quantifier logic presents no additional problem in principle. One would have to introduce individual terms including individual variables, quantified assumption rules $\Delta \Rightarrow_{x_1,\ldots,x_n} \alpha$, and canonical conditions and consequences of formulas $\forall x\alpha(x)$ and $\exists x\alpha(x)$ by defining

the rule $\Rightarrow_x \alpha(x)$ to be an $(\forall x\alpha(x))^-$,
each $\alpha(t)$ for an arbitrary term t to be an $(\exists x\alpha(x))^-$,
each $\alpha(t)$ for an arbitrary term t to be an $(\forall x\alpha(x))^+$,
each rule $\langle\langle\alpha(y)\rangle \Rightarrow_y \beta\rangle \Rightarrow \beta$ for an arbitrary formula β not containing y free to be an $(\exists x\alpha(x))^+$.

The soundness and completeness of intuitionistic quantifier logic can then be proved in a similar way (without $\vee$ and $\exists$ in the I rule approach).

As Prawitz (1971, p. 290) pointed out, in the E rule approach one is even able to handle universal quantification within contexts where the quantifier is not understood in the schematical sense, as e.g., in Peano arithmetic: If numerals *n* are available for every n, we may define each $\alpha(n)$ for arbitrary n to be an $(\forall x\alpha(x))^+$. According to that definition, the induction rules $\langle\alpha(O), \langle\alpha(x)\rangle \Rightarrow_x \alpha(x')\rangle \Rightarrow$ $\Rightarrow \forall x\alpha(x)$ turn out to be valid- in the atomic calculus characterizing Peano arithmetic because for each n there is a derivation of $\alpha(n)$ from $\alpha(O)$, $\langle\alpha(x)\rangle \Rightarrow_x \alpha(x')$ which applies no basic rule. However, unlike Prawitz' sketch for an E rule approach (ibid.), our soundness theorem cannot in that case be transferred to Peano arithmetic. This is due to the fact that our restrictions on the complexity of rules used in k-valid- derivations make it necessary to require normal derivations in $I(A)$ for our soundness theorem which cannot be fully obtained in Peano arithmetic. Another reason is that the important lemma 2.2.2(ii) would not hold for ∀-formulas. The fact that the completeness theorem cannot be upheld follows from Gödel's incompleteness theorem. The inadmissibility of the ω-rule in Peano arithmetic blocks step (iii)a) of the proof of theorem 3.2.

5.2. The Failure of the I Rule Approach for Formulas with ∨.

Lemma 2.1.1(i) is crucial for many succeeding lemmata and for the theorems. It states that the rank of an I rule for α is $|\alpha| + 1$, and that the rank of an E rule with α as its major premiss is $|\alpha| + 1$ if α is not a disjunction or the absurdity. That means that the rank of these rules only depends on the rank of α and not on the rank of any other formula. This no longer holds in the case of ∨E rules

$$\frac{\alpha \vee \beta \quad \begin{matrix}\alpha \\ \gamma\end{matrix} \quad \begin{matrix}\beta \\ \gamma\end{matrix}}{\gamma} \; .$$

Their rank not only depends on the major premiss $\alpha \vee \beta$ but can be arbitrarily high if γ is complex enough. (The same holds for ⊥E, but there it causes no problems because conclusions of applications of ⊥E can be assumed to be atomic.) Thus neither lemma 2.4.6 nor lemma

2.4.7(i) which are important tools in the soundness theorem 4.1 can be proved if $\vee$ is admitted. However, lemma 2.4.7(ii) holds for $I(A)$ instead of $I^{o}(A)$, because in $I(A)$ we do not have any restriction on ranks of applied rules. So the completeness theorem 4.2 can be upheld if $\vee$ is included. Furthermore, lemma 2.4.6 remains true in the presence of $\vee$ for the case that $|\Phi| \leq |\alpha|$, as can easily be checked.

These results can be used to demonstrate that a soundness proof for full intuitionistic logic over A including $\vee$ with respect to validity+ in A is not possible for all A. Consider the rule ρ:

$$\frac{(\alpha \vee \beta) \;\&\; \gamma}{(\alpha \;\&\; \gamma) \vee (\beta \;\&\; \gamma)}$$

over F where α, β, γ are atomic formulas (i.e., sentence letters), which is of rank 3. If ρ were valid+ in A, by the part of lemma 2.4.6 which remains provable, we would have

either $\alpha \vee \beta, \gamma \vdash^{A,+}_{2} \alpha \;\&\; \gamma$

or $\quad \alpha \vee \beta, \gamma \vdash^{A,+}_{2} \beta \;\&\; \gamma$,

thus by the completeness theorem

either $\alpha \vee \beta, \gamma \vdash_{I} \alpha \;\&\; \gamma$

or $\quad \alpha \vee \beta, \gamma \vdash_{I} \beta \;\&\; \gamma$,

which does not hold. ρ is, however, derivable in I by means of the (normal) derivation

$$\dfrac{\dfrac{\dfrac{}{(\alpha \vee \beta) \,\&\, \gamma}}{\alpha \vee \beta} \qquad \dfrac{\dfrac{\overset{}{\dfrac{}{\alpha}}{\scriptstyle(1)} \qquad \dfrac{\dfrac{}{(\alpha \vee \beta) \,\&\, \gamma}}{\gamma}}{\alpha \,\&\, \gamma}}{(\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)} \qquad \dfrac{\dfrac{\dfrac{}{\beta}{\scriptstyle(1)} \qquad \dfrac{\dfrac{}{(\alpha \vee \beta) \,\&\, \gamma}}{\gamma}}{\beta \,\&\, \gamma}}{(\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)}}{(\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)}\,{\scriptstyle(1)}\ .$$

One immediately sees that one cannot get rid of the application of the vE rule

$$\dfrac{\alpha \vee \beta \qquad \overset{\alpha}{(\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)} \qquad \overset{\beta}{(\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)}}{(\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma)}$$

which is of rank 4, even if the premisses of ρ are inferred by I rules (so that one has $\alpha \vee \beta$ and γ as undischarged assumptions).

Since soundness, unlike completeness, is a necessary condition for the adequacy of a semantics, the approach based on I rules definitely fails for full intuitionistic logic. It might be noted that Prawitz (1984, section 5) refers to difficulties of the kind just described as being reasons for preferring the validity of derivations ('arguments' in his terminology) instead of rules as the primary notion: He mentions the problem that the derivation of a premiss of an application of an $\supset$I rule may essentially contain applications of E rules of higher complexity than the conclusion of this application, which is exactly the case if we extend the derivation given above by application of an $\supset$I rule to a derivation of $((\alpha \vee \beta) \,\&\, \gamma) \supset ((\alpha \,\&\, \gamma) \vee (\beta \,\&\, \gamma))$.

Perhaps validity of derivations is the better tool when relying on I inferences, and validity of rules the better tool when relying on E inferences. --

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