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Section 16: History of Logic, Methodology, and Philosophy of Science

GENTZEN-STYLE FEATURES IN FREGE

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As has been recently observed ([4],[5]), there is a striking resemblance between certain features of Frege's formal systems and Gentzen-style formulations of logical calculi, although they are historically unrelated. This is especially significant as Frege-style and Gentzen-style systems are normally considered to be fundamentally different, the former being prototypes of Hilbert-type calculi.

Kutschera [4] showed that the first-order fragment of the system proposed by Frege in his *Grundgesetze der Arithmetik* [2] can be understood as a sequent-style natural deduction system. For the implicational part this result may be put as follows. Consider the Gentzen-style system with the following rules of inference, where in our linear notation the slash '/' denotes an inference line:

$/ \; A, B \Rightarrow A$ with $/ \; A \Rightarrow A$ as a limiting case

$\Gamma \Rightarrow A \; / \; \Gamma' \Rightarrow A$ (Γ' permutation of Γ)

$\Gamma[A \ldots A] \Rightarrow B \; / \; \Gamma[A] \Rightarrow B$ (Contraction of two or more occurrences of A)

$\Gamma, A \Rightarrow B \; / \; \Gamma \Rightarrow A \to B$ ($\to$ introduction)

$\Gamma \Rightarrow A \quad \Delta \Rightarrow A \to B \; / \; \Delta, \Gamma \Rightarrow B$ ($\to$ elimination)

Let the *Frege counterpart* of a sequent $A_1, \ldots, A_n \Rightarrow A$ be the implicational formula $A_1 \to (\ldots \to (A_n \to A) \ldots)$ (with A being the Frege counterpart of $\Rightarrow A$) — of course to be written two-dimensionally in Frege's original notation. Then any derivation in the Gentzen-style calculus yields a derivation in Frege's system. We just have to replace every sequent with its Frege counterpart und to delete all applications of ($\to$ introduction) (whose premiss and conclusion is translated into the same implicational formula).

Conversely, by writing implicational formulas $A_1 \to (\ldots \to (A_n \to A) \ldots)$ as sequents $A_1, \ldots, A_i \Rightarrow A_{i+1} \to (\ldots \to (A_n \to A) \ldots)$ (with $\Rightarrow A_1 \to (\ldots \to (A_n \to A) \ldots)$ being a limiting case), a derivation in the above Gentzen-style system is obtained from any derivation in Frege's system. To cope with the ambiguity of splitting up an implicational formula at a particular place A_i, it may be necessary to insert applications of ($\to$ introduction), as well as applications of ($\to$ elimination) with the left (minor) premiss being of the form $A \Rightarrow A$.

This paper discusses whether this resemblance is just a technical coincidence with no deeper bearing on the notion of a logical system, or whether it shows that Frege anticipated certain ideas later developed by Gentzen [3].

In spite of the fact that in Frege there is no syntactical distinction between implications and sequents, and that Frege repeatedly and explicitly rejects the idea of a conceptual difference between assumptions in proofs and hypotheses of implications, it can be argued that to some extent he is aware of the Gentzen-style features mentioned. Actually, his metalinguistic distinction between the 'Oberglied' (= *succedens*) and the 'Unterglieder' (= *antecedentia*) of an implicational formula crucially enters his formulation of the inference rules in [2]. Thus Frege's formalism may appropriately be called a 'metalinguistically specified sequent system'. To give an analogy, we may refer to certain formalisms considered by Schütte [6] which are not sequent calculi in the syntactic sense, but are specified as sequent systems by means of a metalinguistic classification of formula parts as 'positive' or 'negative' .

This view is further supported by the fact that Frege's distinction between 'Oberglied' and 'Unterglieder' is drawn only at the uppermost level of formula construction, but never at the level of embedded implications. All this adds to the strong *prima facie* plausibility our interpretation gains from Frege's explicit choice of structural rules ('Vertauschung', 'Zusammenziehung') as primitive rules of inference in the *Grundgesetze*, which renders this system fundamentally different from the *Begriffsschrift* [1] system.

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