Meaning explanations and dialogues

Ansten Klev
Philosophical Institute, Czech Academy of Sciences

Proofs, Arguments and Dialogues
Tübingen, 9 August 2022

Overview

- This lecture will be an introduction to the idea of a meaningful formal system – with emphasis on *meaningful*.
- The idea will be illustrated by means of Martin-Löf's type theory, a formal system for which detailed meaning explanations have been developed.
- We will test the idea by seeing how it makes possible the justification of the rules of inference in this formal system.
- Dialogues will be introduced towards the end,
  1. as an alternative way of formulating the meaning explanations;
  2. as a way of spelling out the notion of assertoric knowledge, needed in the definition of the validity of an inference.

Proof

From the verb *to prove* (Oxford English Dictionary):

> *4. To establish as true; to make certain; to demonstrate the truth of by evidence or argument.*
>
> *. . .*
>
> *1. To put (a person or thing) to the test; to test or assess the genuineness or qualities of. Now rare in general use.*

For 1, compare other Germanic languages (and Italian *prova*?) and, for example, "proofs" of an article.

Argument

Cicero:

> *we may define argument as a reasoning that lends belief to a doubtful issue [ratio, quae rei dubiae facit fidem]*

Oxford English Dictionary:

> *2. A statement or fact advanced for the purpose of influencing the mind; a reason urged in support of a proposition.*
>
> *...*
>
> *4. A connected series of statements or reasons intended to establish a position (and, hence, to refute the opposite); a process of reasoning; argumentation.*

Demonstration

Oxford English Dictionary:

> *1. The action, process, or fact of establishing the truth of a proposition or theory by reasoning or deduction...; an argument or sequence of propositions showing that a conclusion, theorem, etc., is a necessary consequence of axioms or previously accepted statements.*

Ultimately from the Greek *apodeixis*, verb *apodeiknymi*, to show. German *Beweis*.

Terminological stipulation

$$\begin{aligned} & \text{Demonstration} \\ = \quad & \text{Deductive argument} \\ = \quad & \text{Sequence of valid inferences} \\ & \text{from immediately evident starting points} \end{aligned}$$

Serves to make its conclusion evident.

Inference

$$\frac{J_1 \; \dots \; J_n}{J} \qquad \text{(Inf)}$$

The letter J here stands for "judgement".

That the inference is *valid* means that

- the conclusion J can be known under the assumption that the premisses $J_1, \dots, J_n$ are known.

Contrast this with the *holding* of a consequence:

- the proposition A is true provided the propositions $A_1, \dots, A_n$ are all true.

Two central tasks of formal logic

1. Describe forms/patterns of inference.
    - These patterns are recorded as rules of inference.
2. Justify these rules of inference. Show that they capture patterns of *valid* inference.

Some systems of formal logic

- Aristotle's syllogistics
- Leibniz's equational logic
- Frege's ideography
- Modern (metamathematical) predicate logic
- Martin-Löf's type theory

Modern predicate logic I

A "language" $\mathscr{L}$ is an inductively defined set of formulas, φ.

- Formulas are formal objects, like numbers.
- They are *not* expressions – they cannot be used to express anything.

A "semantics" or "interpretation" of $\mathscr{L}$ is a mapping of $\mathscr{L}$ into a mathematical structure $\mathfrak{M}$.

Then one defines a relation, $\mathfrak{M} \models \varphi$, read $\mathfrak{M}$ satisfies φ.

This definition does not give meaning to φ, but to "$\mathfrak{M} \models \varphi$".

If $\mathfrak{M}$ is a set-theoretic model, then "$\mathfrak{M} \models \varphi$" is a set-theoretic proposition.

Modern predicate logic II

An "inference",

$$\frac{\psi_1, \ldots, \psi_n}{\varphi}$$

is "valid" if for every $\mathfrak{M}$,

$\mathfrak{M} \models \varphi$ is true provided $\mathfrak{M} \models \psi_1, \ldots \mathfrak{M} \models \psi_n$ are all true

Side remark: this is a reduction of *inference* to *consequence*.
Göran Sundholm has called this "the Bolzano reduction".
It is taken for granted in most of current logic and philosophy.

The language of set theory

Two forms of atomic formula: $x \in y$ and $x = y$.

How do we give meaning to this language?

We make stipulations regarding the notion of set.

We spell out a *conception* of set, e.g., the iterative conception, answering the questions

- What is a set?
- When are sets equal?
- What does it mean to be an element of a set?

Martin-Löf type theory

External characteristics:

- Developed around 1970 as a foundation for constructive mathematics.
- Later (from 1980s) it has found important applications in computer science and linguistics.
- Homotopy type theory (HoTT) is, formally speaking, just an extension of Martin-Löf type theory with some new axioms.

Internal characteristics:

- Synthesizes ideas from constructive mathematics and logic, typed lambda-calculus, and proof theory.
- Based on the *principle of types*: every object is typed.
- It is equipped with precise meaning explanations.

Portrait

Per Martin-Löf

The language

The unit of expression is called a *judgement*.

For simplicity, we will concentrate on two forms of judgement:

$$A : \textbf{type}$$
$$a : A$$

These are read:

A is a type
a is an element of the type A

We shall concentrate on types of individuals, hence

$$\text{type} = \text{type of individuals}$$

Type theory and logic

It may seem restrictive only to be allowed to say that something is a type, $A : \mathbf{type}$, or an element of a type, $a : A$.

The Curry–Howard correspondence gives an embedding of full (constructive) predicate logic into type theory.

$$\begin{aligned} \text{type} &= \text{proposition} \\ \text{element of } A &= \text{proof of } A \end{aligned}$$

In other words, we read

$$\begin{array}{ccc} A : \mathbf{type} & \text{as} & A \text{ is a proposition} \\ a : A & \text{as} & a \text{ is a proof of } A \end{array}$$

Meaning explanations

- We are not interested merely in a formal system that we can say something *about* (metalogic).
- Instead we want a language that we can use to say something with.
- The language of set theory is often assumed to serve such a role.
- Type theory will be another such language.
- Martin-Löf's meaning explanations for his type theory are much more detailed than anything similar provided for classical set theory.

Some questions

Meaning explanations should provide answers to the following two questions (among many others):

- What is a type?
- What does it mean to be an element of a type?

There are parallel questions of identity:

- What is it for A to be the same type as B?
- What is it for a to be the same element of A as a'?

We shall not deal with these here.

Answers

We transform the questions into

What must I know in order to have the right to assert

- A : **type**?

I must know how the canonical elements of A are formed (and how identical canonical elements are formed).

- $a : A$?

I must know that a evaluates to a canonical element of A.

Evaluation

By evaluation we understand *definitional reduction*.

$$2 \underset{\text{Def of 2}}{\longrightarrow} \mathbf{s}(1) \underset{\text{Def of 1}}{\longrightarrow} \mathbf{s}(\mathbf{s}(0))$$

$$1+1 \underset{\text{Def of 1}}{\longrightarrow} 1+\mathbf{s}(0) \underset{\text{Def of 1}}{\longrightarrow} \mathbf{s}(0)+\mathbf{s}(0) \underset{\text{Def of +}}{\longrightarrow} \mathbf{s}(\mathbf{s}(0)+0) \underset{\text{Def of +}}{\longrightarrow} \mathbf{s}(\mathbf{s}(0))$$

Non-mathematical example:

human $\longrightarrow$ mortal rational animal

$\longrightarrow$ mortal rational percipient living body

$\longrightarrow$ mortal rational percipient living corporeal substance

Canonical element

- A canonical element is an endpoint of evaluation.
- Since evaluation is definitional reduction, a canonical element is an element given in fully primitive notation.
  - This is a half-truth: lazy evaluation.

In a school class, a teacher asks, What is 7×5?

In one sense, 7×5 is a correct answer, but the teacher wants the same number canonically given: 35.

Generation of canonical elements

By inductive definition.

Paradigm:

$$0 : \mathbf{N} \qquad \frac{n : \mathbf{N}}{\mathbf{s}(n) : \mathbf{N}}$$

Also an inductive definition (degenerate):

$$\mathrm{t} : \mathbf{bool} \qquad \mathrm{f} : \mathbf{bool}$$

Also:

$$\frac{a : A \qquad b : B}{\langle a, b \rangle : A \times B}$$

Type-forming operators: examples

nullary: **N**, **bool**, $\emptyset$ ($\perp$)

binary:

$A \times B$	$A + B$	$A \to B$
$A \wedge B$	$A \vee B$	$A \supset B$

dependently typed:

$\mathbf{Id}(A, a, b)$	$(\Pi x : A)B$
$a =_A b$	$(\forall x : A)B$

Rules

With each type-forming operator there are associated four sorts of rule.

- **Formation rule** describes the behaviour of the type-forming operator in question – how a type is formed by means of it.
- **Introduction rules** describe how the canonical elements of the type are formed.
- **Elimination rules** stipulate the existence of certain functions going out from the type.
- **Equality rules** give the definitional equations for such functions.

Example: binary product (conjunction)

$$\frac{A : \textbf{type} \qquad B : \textbf{type}}{A \times B : \textbf{type}} \qquad (\times\text{-form})$$

$$\frac{a : A \qquad b : B}{\langle a, b \rangle : A \times B} \qquad (\times\text{-intro})$$

$$\frac{c : A \times B}{\mathsf{fst}(c) : A} \qquad \frac{c : A \times B}{\mathsf{snd}(c) : B} \qquad (\times\text{-elim})$$

$$\begin{aligned} \mathsf{fst}(\langle a, b \rangle) &= a : A \\ \mathsf{snd}(\langle a, b \rangle) &= b : B \end{aligned} \qquad (\times\text{-eq})$$

The justificatory status of the rules

- Introduction and equality rules are stipulations, and a question of justification does not arise for them.
  - It is, however, not so that anything goes: we must check that they serve their intended role as definitions.
- Formation and elimination rules, by contrast, are in need of justification.
  - A formation rule asserts that a certain type exists (under the assumption that certain other types exist). We must show how its canonical elements are formed.
  - An elimination rule introduces a non-canonical element of a type. We must show how it is evaluated to a canonical element.

Roughly:

- Formation rules are justified by introduction rules.
- Elimination rules are justified by equality rules.

Example: justification of $\times$-formation

$$\frac{A : \mathbf{type} \qquad B : \mathbf{type}}{A \times B : \mathbf{type}} \quad (\times\text{-form})$$

- Assume that we know the premisses, $A : \mathbf{type}$ and $B : \mathbf{type}$.
- We then understand judgements of the form $a : A$ and $b : B$.
- Hence we also understand $\times$-introduction, regarded as a stipulation:

$$\frac{a : A \qquad b : B}{\langle a, b \rangle : A \times B} \quad (\times\text{-intro})$$

- This stipulation shows how the canonical elements of $A \times B$ are formed.

Example: justification of $\times$-elimination

$$\frac{c : A \times B}{\mathrm{fst}(c) : A} \qquad (\times\text{-elim})$$

- Assume that we know the premiss, $c : A \times B$.
- We then know an a in A and a b in B such that $c \Rightarrow \langle a, b \rangle : A \times B$.
- By definition, we have

$$\mathrm{fst}(c) = \mathrm{fst}(\langle a, b \rangle) = a : A$$

- Since a is an element of A, we have $a \Rightarrow a' : A$, for a canonical element a' of A.
- Since evaluation is definitional reduction, also $\mathrm{fst}(c) \Rightarrow a' : A$.

Justifying the rules of inference

In Martin-Löf type theory, all rules of inference are justified by reference to the meaning explanations.

- A stipulatory rule (e.g. introduction or equality rule) is justified by the role that such a rule plays according to the meaning explanations.
- A postulatory rule (e.g. formation or elimination rule) is justified by the meaning given to the judgements involved as premisses and conclusion according to the meaning explanations.

An explanatory circle

Recall the definition of the validity of an inference,

$$\frac{J_1 \ \ldots \ J_n}{J}$$

- The conclusion J can be known under the assumption that the premisses $J_1, \ldots, J_n$ are known.

In mathematics and logic, "knowing J" typically means "having demonstrated J":

- the conclusion J can be demonstrated under the assumption that the premisses $J_1, \ldots, J_n$ have been demonstrated.

We defined a demonstration to be a sequence of valid inferences.

We are caught in an explanatory circle:

demonstration refers to *valid inference* refers to *demonstration*

Enter dialogues

In recent years, Martin-Löf has discovered that the notion of dialogue is useful for avoiding the circle.

More precisely, the notion of dialogue can be used to articulate an alternative notion of "knowing J" that is weak enough to avoid the circle, but strong enough to support the definition of validity of inference.

Knowing J as knowing how

- We stipulate that the content of a judgement is a task.
- A judgement is correct if the agent making it *knows how* to perform (do, fulfil) the task that constitutes its content.

A dialogical view of judgement/assertion:

- If I make a judgement with content C, and an interlocutor challenges my assertion, then I am obliged to do C.

$$\frac{\vdash C \qquad ?\,C}{C \text{ done}}$$

Assertoric knowledge

Recall from previous slide:

- A judgement is correct if the agent making it *knows how* to perform (do, fulfil) the task that constitutes its content.

We call this knowledge-how *assertoric knowledge*.

Novel definition of the validity of an inference:

- The conclusion can be assertorically known under the assumption that all of the premisses are assertorically known.

The explanatory circle is now avoided, since we do not appeal to the notion of demonstration.

Dialogue rules

The dialogue structure

$$\frac{\vdash C \qquad ?\, C}{C \text{ done}}$$

spells out the assertoric knowledge conveyed by the judgement $\vdash C$.

Doing C might well consist in making further judgements.

Examples:

$$\cfrac{c : A \times B}{\begin{array}{c}\langle a, b \rangle : A \times B \\ c \Rightarrow \langle a, b \rangle : A \times B\end{array}}\,? \qquad\qquad \cfrac{\langle a, b \rangle : A \times B}{\begin{array}{c}a : A \\ b : B\end{array}}\,?$$

Similar rules are given for each type-forming operator.

Rules, plays, strategies

- It is the dialogue rules that primarily matters here.
  - They provide a formal presentation of the meaning explanations.
  - The dialogical format shows that understanding J is understanding what it takes assertorically to know J.
- Plays are important because they illustrate the rules in action.
- The notion of a (winning) strategy seems to play no role at all.